TORSION OF PRISMATIC MEMBERS OF NON-LINEAR MATERIALS

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V.K. KAPUR

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CERTIFICATE

This is to certify that the work presented in this thesis has been carried out by Mr. V.K. Kapur under our supervision and has not been submitted elsewhere for a degree.

(Dr. A.H. Shah)
Assistant Professor
Dept. of Civil Engineering
I.I.T. Kanpur

(Dr. J.K. Sridhar Rao)
Assistant Professor
Dept. of Civil Engineering
I.I.T. Kanpur

ABSTRACT

This dissertation deals with the energy method for the torsional analysis of prismatic members which exhibit a linear or non-linear stress-strain behaviour. Various stress-strain relations in nondimensional form are used as idealisation of material properties so that these can be used to cover wide range of concretes and other structural materials.

A stress function satisfying the boundary conditions and having unknown coefficients is assumed. The unknown coefficients in the stress function are determined by minimising the complementary potential. Results obtained numerically, are presented for rectangular and circular cross-section for different parameters in the generalised Ramberg-Osgood relation, thereby covering materials like concrete, steel, aluminium, magnesium and reinforced plastics. Comparison of results with the existing experimental and analytical results is made. Finally a methodology for ultimate strength theory using hyperbolic sine relation for stress strain expression is presented.

NOTATIONS

A : area of cross-section

a : radius of circular cross-sections

a,b : dimensions of rectangular cross-section, width = 2a, depth = 2b.

A, B, C, A_i, A_{ij}: Unknown Coefficients

E : modulas of elasticity in tension

E_p : modulas of elasticity in tension for plastic strain.

e, ϵ : Strain

f ; Assumed form of function for strain

f'_c : cylinder compressive strength of concrete

f'_t : tensile strength of concrete

G, G_o ; Shear modulas

G_p : Shear modulas for plastic portion

h : 2a

K : b/a

K_1 : $\dfrac{\bar{T}}{8a^2bf'_t}$

J : polar moment of interia of test specimen

i,j,k,r,p,q: 1,2,3

L : Length of prismatic member

r : one of the parameters in Ramberg-Osgood function

m : r+1/2

s : arc parameter

R : In the small eighbourhood of the solution

T : Torque

$\bar{T}$	: Ultimate Torque
u, v, w	: displacement components
U_c	: total complementary energy in the member
U_{co}	: total complementary energy per unit-volume
x, y, z	: coordinates
x, y	: x/a, y/b
W_e	: Total complementary work done by the external torque.
Π	: complementary potential energy
ϕ	: stress function
θ	: twist per unit length
$\bar{\theta}$	: Ultimate twist per unit length
ψ	: warping function
γ_{xz} γ_{yz}	: shear strain components
	: shear strain resultant
E, P	: shear strain resultant for elastic and plastic portion respectively
$\bar{\gamma}$	: Ultimate shear strain of material
τ_{xz} τ_{yz}	: shear stress resultant shear stress resultant
	: shear stress resultant
τ_o, γ_o	: non-dimensionalising shear stress and shear strain
δ	: parameter of Ramberg-Osgood function
$\bar{\epsilon}$, $\bar{\sigma}$	: maximum normal tensile strain and tensile stress
	: Poisson's ratio
α	: $\bar{\tau}/f'_t$
β	: $f'_t / \overline{f'_c}$

γ^*, τ^*	:	γ/γ_o , τ/τ_o
τ^*_{max}	:	τ_{max}/τ_o
T^*	:	$\dfrac{T}{\tau_o \times \text{area} \times h}$
θ^*	:	$\theta h/\gamma_o$

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1. INTRODUCTION

Methods exist for the analysis of torsion of prismatic members of isotropic, homogeneous materials which are either linearly elastic or rigid perfectly plastic. Structural materials in general, are neither elastic nor perfectly plastic throughout the range of behaviour either in pure compression or pure tension. This simple fact underlines the necessity and importance of analytical work for non-linear elastic materials in torsion.

In the past such a work was not considered feasible because of the difficulties the material non-linearity introduces. It makes the resulting simultaneous equations or the governing differential equations non-linear in nature and at that time these non-linear equations were rather difficult to solve. So, naturally for design work etc., resort was taken to the then existing results by assuming the material to be either elastic or perfectly plastic. But now, with the advent of electronic computers along with the vast development of numerical techniques, hurdles created by non-linearity has altogether been eliminated. Though, inspite of all this, little effort has been made in this field of inelastic analysis for torsion but definitely research is gaining momentum in this field The present work is a step in this direction.

1.1 Present Method and Objectives

An attempt is made herein to develop a non-linear elastic analysis for prismatic members in torsion for short term loading using energy approach, for general idealisation of inelastic material behaviour. The objectives of an inelastic analytical method are two-fold:

1. To get an analytical understanding of the behaviour of prismatic members subject to torsion, from begining to failure.
2. The knowledge of inelastic behaviour helps in the understanding of shear stress-strain behaviour from uniaxial tension, compression and torsion tests.

In this analytical investigation various stress strains curves are used, such as: parabolic and polynomial expressions, Ramberg-Osgood function, Sinh curve. Any of these curve can be used to represent the shear stress-strain behaviour of the given material. A complementary energy approach is used wherein the stress function ϕ is assumed to satisfy the boundary conditions. The unknown parameters in the stress function are deteemined by minimising the total comolementary energyc potential for an assumed value of the twist θ . Then the twisting moment and the maximum resulting shear stress τ for this value of θ are determined. Thus theoritical plots of T Vs θ and τ_{max} Vs θ curves are obtained. These are compared with the available analytical and experimental results.

1.2 Review of Previous Work

Coulomb(1)* was first to give an exact solution of the torsional problem for a circular bar of homogeneous, isotropic, elastic material. The main assumption behind such a solution was that plane section normal to the longitudinal axis of twist always remain plane. Later when Navier(2) tried to extend the above theory to noncircular sections he ended with contradiction with the boundary conditions. However, such an attempt made it clear that plane sections normal to the longitudinal axis of twist in case of nonculular cross-sections does not remain plane after the application of torque. Saint Venant(3) guided by Navier's conclusion assumed both rotation and warping of section and put forwarded the so called semi-inverse method for the solution of torsion problem for prismatic members. Afterwards mathematical solutions for the torsion problem for different cross-sections have been obtained. Seth(4) gave the solution of T-beam with infinite flanges by using complex variable approach whereby a T-shaped section was mapped on a unit half circle by using the Swatz Chritoftel transformation.

Iyeangar(5) provided the rigrous solution for T,L and I sections. By dividing the flanged sections into two rectangles i.e. web rectangle and flange rectangle, he

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*Number in brackets designate references at the end.

assumed two stress functions for these two rectangles. These stress function were determined in such a way that besides satisfying the boundary conditions on the respective rectangle, these were required to satisfy the continuity conditions at the junction.

For some of the sections, the mathematical analysis became so complicated that it gave rise to a new method of solving torsion problem. This was to find mathematical analogy between the torsion problem and the behaviour of some other physical problem which is easy to visualize and solve. One such analogy was given by Prandtl[6] and is known as Membrame Analogy. This is based on the mathematical similarity between the torsion problem and the behaviour of a uniformly stretched elastic membrane subjected to uniform lateral pressure. The membrane has the same shape as the cross-section and its edges are restrained. The deflection of such a membrane give rise to a differential equation similar to the governing differential equation in torsion problem. And hence the deflection function in the above problem corresponds to the stress function in the original problem. Such an analogy is useful in the experimental determination of stress function and stress distribution.

Based on the membrane analogy, soap films as membranes have also been in use for the direct measurement of stresses. Two holes are made in a plate, one of

the same cross-section as under investigation and the other a circular hole of predetermined diameter. Films on both these holes are applied with the same lateral pressure. By measuring the slopes of two films, comparison of the stresses of the given member with the circular bar for the same unit twist can be made. The ratio of the volume under the two films give the corresponding torque ratio.

Besides this, there are number of hydrodynamical analogies. Kelvin(7) suggested the similarity between stress function and stream function of the irrotational motion of an ideal fluid in a vessel of the same cross-section as the twisted bar.

Boussinesq(8) showed the similarity between determination of stress function and Velocities in a laminar motion of viscous fluid in a tube of the same cross-section. All these hydrodynamical analogies may not be important for the actual solution of torsion problem, but these definitely help one to draw certain important conclustins which otherwise are difficult to draw. One such conclusion is that at the projecting corners of a cross-section of a twisted bar the shearing stresses become zero whereas at the reentrant corner these stresses become theoritically infinitely large.

Buchanan (9) for the first time studied the torsion problem of prismatic members of isotropic, homogeneous

material with non-linear stress-strain relation. This non-linear relation between shearing stress and shearing strain was determined experimentally from torsion tests on circular beams. Such a method of obtaining shear stress strain curve gave good results as confirmed by the comparison of computed and experimental results.

Two series of tests were conducted. One was to study the failure mechanism and the cracking pattern of rectangular and T-shaped cross-section for various combinations of longitudinal and transverse reinforcement From these tests it was concluded that failure of reinforced T beams depends upon the type and location of reinforcement.

In the second series of test, three circular, one rectangular and one T-shaped cross-section without reinforcement were loaded to failure in pure torsion to obtain torque-rotation data. The theoritical analysis employed is nothing but the modification of Saint-Venants theory to take into account the material nonlinearily. Shear modulas G, which is constant for a linear case now becomes a function of the stress level in otherwords a function of the coordinates (x,y) in the cross-section. Finite difference technique was employed to solve the governing differential equation. Each cross-section was divided into small grids and a difference equation was written for each grid point. By solving the simultaneous

equations. ϕ was obtained for a particular value of θ. The θ is then increased by a small amount and the corresponding value of shear stress and torque were determined. Comparison with experimental results shows good correlation.

In all these test, model beams of plaster were used in place of structural concrete prototypes. This was done with the view to reduce the curing time but care was taken to take a proper mix of plaster so as to represent the stress strain behaviour of concrete prototype.

From the above study, Buchanan concluded that for structural concrete which does not have plastic plateau on its stress strain curve, strain hardened distribution is more suited than plastic stress distribution near failure. According to him, there is very little difference in the computed torque capacity for either ultimate stress distribution, which possibly explain why plastic stress distribution have given satisfactory results in the past.

Smith and Sidebottom(10) made use of principle of minimum complementary energy potential for solving the torsion problem for a material with non-linear stress strain law. Shear stress-strain relation was represented by hyperbolic Sinh function in terms of normal (tensile) stress strains. This was done by making use of the Von Mises Octahedral shear stress theory to convert uniaxial state of stress by multiaxial state of stress. The stress function

$\emptyset$ satisfying the boundary conditions was obtained by minimising the complementary potential w.r.t. each of the unknown parameters in the stress function. Knowing $\emptyset$, shear stress and torque for different value of twist were obtained. However such a solution is approximate unless the number of functions satisfying the boundary conditions in the stress function tends to infinity. Taking few such functions sufficiently accurate result were obtained for rectangular and trianguar cross-sections. For circular case, exact results have been given.

Shah, Setlur & Chatterjee[11] have applied the finite element method. The continuous two dimensional problem was discretized by dividing the cross-section into a finite number of triangular elements. Complementary potential energy of the member expressed in terms of the nodal stress function values and angle of twist, was minimised and thereby obtaining a set of non-linear algebraic equations. These were solved numerically to give $\emptyset$ for one value of unit twist. Thus plots of torque Vs twist for circular, rectangular and L cross-section for different material properties were obtained. Comparison of results with existing results show good agreement.

Design of R.C.C. structures for torsion is still considered a difficult problem in structures of ordinary proportion. With the increase in use of monolithic structures, it has now become an important problem. So lot of

research is being done to obtain torsional strength of concrete members, which is essentially a problem concerning non-linear material. The main problem with concrete is that it is weak in tension than in shear with the result that it fails in tension much before a failure due to shear takes place. The approximate relation between compressive, tensile and shear strength of concrete is in the ratio of 10:1:2 [12]. Besides this other difficulties associated with it are, determination of maximum tensile stress and its variation and problem of arbitrary cross-section. Perhaps these are some of the points which justify the seprate treatment that has been given to it besides other structural materials.

Plain concrete is rarely used where torque is likely to be applied. But tests, discussed later, show that addition of reinforcement does not change the behaviour of members under torsion, atleast up to the first crack. Knowledge of the behaviour of the section upto first crack is also important from practical considerations like, serviceability, deflection and rotation. Moreover the increase in torsional strength due to reinforcement is hardly of the order of 15% . Tests by Hsu [13] showed that rectangular sections of plain concrete fails as soon as the first crack appears. Torque and twist curve show a linear relation upto approximately 80% of the torsional capacity before begining to show slight non-linear behaviour due to the inelastic redistribution of stress near

failure.

In flanged sections of plain concrete in pure torsion the failure is not so sudden. Zia[14] observed that first crack appears in the web and the beam continues to carry more torque until the crack propagates through flanges and the member fails.

Hsu on the basis of experiments concluded that rectangular concrete beams in pure torsion actually fail by bending. This was based on the criteria that failure is reached when the tensile stresses indueed by a 45° bending component on the under face of the rectangle reaches a reduced modulas of rupture . Making use of equilibrium at failure, he gave a mathematical relation for ultimate torque, valid for rectangular cross-section. For other flanged rections he suggested the algebraic sum of torque for each of the divided rectangle. The torque so given is on conservative side and has been supported by experimental results.

Zia[14] found that basically the torsion theories for plain concrete can be expressed in a basic form as

$$\tau = \frac{T}{KAD}$$

where τ = maximum torsional stress

T = torque

A = Cross-sectional area

D = diameter of the inscribed circle

The summary of various theories in the tabular form is given in Table 1 values of K_e, K_p depend on the ratio of h/b and varies between 0.2 to 0.5. $(\frac{PD}{4A})$ is the shape factor, where P represents perimeter of the section. And b_i and h_i are width and depth, respectively, of the component rectangles.

Research on R.C.C. members subjected to torsion by number of authours(13,14-20) showed that the behaviour of type of specimen upto cracking is nearly the same regardless of whether it is plain, reinforced longitudinally or transversely or continuously bound by a helix or hoops and longitudinal bars. The specimens behaved classically upto cracking and the cracking torque appear to depend almost entirely on the geometry of the cross-section and the concrete strength and little on the amount of reinforcement present.

Various theories have been proposed to predict the ultimate torque of R.C.C. members depending upon the assumed mode of failure. Rausch, Anderson and cowan based their theories on diagonal tension failure and assumed elastic analysis to be valid. Lessig(21) considered the combined resistance of concrete and reinforcement and redistribution of stresses and cleared some of the points which otherwise could not be explained by elastic theory. His main contention was that after firstcracking, elastic theory is no longer valid.

Table 1

Summary of Torsion Theories for plain concrete

Type of theory	Assumption on stress strain curves for concrete in tension	Circular section of diameter D	Rectangular section of width b and depth h	T-, I-L- Section
Elastic	Linearly elastic	$\tau = \frac{16}{\pi D^3}$	$\tau = \frac{T}{K_e b^2 h}$	$\tau = \frac{3T}{\sum b_i^3 h_i}$
Miyamto	Second degree parabolic	$\tau = \frac{14T}{D^3 \pi}$	Not given	Not given
Turner and Davies	$f_t = 244\text{-}450 \times 10^6 (0.000067 - \epsilon_t)^{1.5}$	$= \frac{13.6}{D^3 \pi}$	$\tau = \frac{0.854 (\frac{PD}{4A}) T}{K_e b^2 h}$	$\tau = (\frac{PD}{4A}) T / \sum K_e b_i^2 h_i$
Marshall and Tembe	$\epsilon_t = \frac{(f_t)^m}{E}$, m= 1.1 to 1.16	$\tau = \frac{(12 + \frac{4}{m}) T}{\pi D^3}$	Not given	Not given
Plastic	Fully Plastic	$\tau = \frac{12T}{\pi D^3}$	$\tau = \frac{T}{K_p b^2 h}$	$\tau = \frac{2T}{\sum b_i^2 h_i}$

Semi Plastic

Realizing the importance of reinforced concrete members subjected to combined torsion and bending or shear or all the three, number of investigators[21-25] have studied the interaction, behaviour and failure mechanisms R.C.C. members subjected to these loadings. Number of expressions have been advanced to calculate the bending moment and ultimate torque. These are based on the equilibrium of forces required just prior to failure mechanism.

Recently, ACI committee on Torsion have advanced some tentative recommendations to be incorporated in the 1970 ACI Building code. In a recent paper, HSu & Kemp[26] have discussed the background behind such recommendations for plain, reinforced and prestressed concrete members subjected to pure torsion. The interaction of torsion with shear and bending have also been discussed.

1.3 Scope: The main utility of present work lies in its application to the practical problems. In the torsion design of machine parts, components of aerospace structures and concrete beams, the present analysis assume greater importance since materials used for these are generally non-linear. However, in order to make full use of present approach, further investigations on flanged & other practical sections including hollow sections statistical treatment of non-homogenity, effect of reinforcement etc. need to be undertaken. Also, rational theories of inelastic behaviour and failures under combined stresses which will help in a clear understanding of shear failures, behaviour under-combined torsion, bending and shear canalso be developed.

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2. STRESS-STRAIN BEHAVIOUR OF STRUCTURAL MATERIALS

2.1 Introduction

In this chapter, various analytical expressions for stress-strain curves are discussed briefly. Out of these few are used as idealisation of stress strain behaviour of wide range of structural materials. Emphasis is laid on materials having non-linear stress strain behaviour. Finally by making use of von Mises Octahedral shear stress theory and tensile stress strain curves, shear stress strain curves are obtained for few structural materials. These curves are idealised by Ramberg-Osgood function and the values of the parameters are given so that these can be used in predicting the behaviour of such materials under torque.

2.2 Preliminaries

In literature number of expressions have been proposed to represent analytically the stress-strain behaviour of materials. This approximation of stress- strain curves with analytical expressions has received much attention mainly because it unifies material behaviour in general by one theory, just like Hooke's law in Elasticity. However, many difficulties come in the way in this sort of representation. These arise because we want these expressions to be simple enough in order to be of practical use and at the same time like to represent the given material as accurately as possible. The following considerations give a good guide for such a selection

(i) The analytical expression should be simple and contain easily determinable parameters. The number of parameters should be sufficient to cover the stress strain behaviour of wide Variety of materials. Two parameters are regarded as the most practical.

(ii) The function should be single valued monotonically increasing up to the ultimate stress and continuous. It should pass through the origin and have a slope there equal to the modulas of elasticity. And should give physically possible values of stress for all possible values of strain or viceversa.

2.3 Analytical Expressions

Polynomial expressions for experimental curves are a common practice. Many of the expressions proposed are particular cases of the general expressions, given by

$$e = \sigma + \beta\sigma^2 + \gamma\sigma^3 + \delta\sigma^4 + \ldots \qquad (2.1)$$

$$\sigma = e + be^2 + Ce^3 + de^4 + \ldots\ldots \qquad (2.2)$$

where e is the strain $\sigma = \frac{s}{E}$ is the ratio of the stress, s, to the modulas of elasticity, E.

Some of the proposed relation(27) with remarks about the suitability in use are given in the tabular form in Table 1.

Table 1

Analytical expressions for stress-strain relations

Name of the proposer	Relation proposed	Test for a good fit, if any straight line curve for σ Vs e	Remarks
1. Hooke	$e = \sigma$		Particular case of (2.1) though so much in use, it is true only for small values of strains.
2. Bulffingeni	$e = K\sigma^{n}$	$\log e$ Vs $\log K + n \log \sigma$	Slope at the origin $\neq E$, unless K=n=1.
3. Riccati	$\sigma = Ke^{-\frac{1}{e'}}$ $\sigma = (\frac{1}{a})(e^{ae'/1+e'}-1)$ e' = base of natural log.		Contain only one parameter and so cann't cover wide variety of materials Slope of the first one is zero at the origin.
4. Gerstner	$\sigma = e+be^2$		Contain one parameter and hence of limited use. Negative values of for $e > -1/b$.
5. Poncelet	$e = [1+\beta(e^{\alpha\sigma}-1)]\sigma$		Not practicable as parameters not esily determinable.
6. Wertheim	$e^2 = \alpha\sigma + \beta\sigma^2$		Slope is zero at the origin.
7. Hodge Kinson	$\sigma = e+be^2+ce^3+de^4$		Particular case of (2.2) contain more than desirable no. of parameters. Negative values of for certain range of e.

8. Cox	$e = \sigma/(1+\alpha\sigma)$	$(\frac{1}{\sigma})\ [(\frac{e}{\sigma}, -1)]$ vs σ	One parameter and so of limited use.
	$e = \sigma + \beta\sigma^2 + \gamma\sigma^3$		Particular cases of 2.1 & 2.2
	$\sigma = e + be^2 + ce^3$	$(1/e)\ [(\sigma/e) - 1]$ Vs e	Negative values of σ or e for certain range of e or σ
9. Imbert	$\sigma = e(\frac{1}{\alpha})\ (e^{\alpha\sigma} - 1)$		Contains one parameter and so of limited use
10. Hartig	$\sigma = (1/a)(e^{ae} - 1)$ $\sigma = \left(\frac{e}{1-e}\right) e^{ae}$		Contains one parameter and so of limited use
11. Schule	$\sigma = Le^m + be^2$		Slope at the origin $\neq$ E, unless L=m=1, in which case reduces to Gerstner's formula.
12. Prager	$\sigma = ae + b \tanh(1+a/b)e$		Parameters not easily determinable for a best fit otherwise widely applicable.
13. Holmquist & Nadai	$e = \sigma\ ,\quad \sigma \leq \sigma_Y$ $e = e + K(\sigma - \sigma_Y)^n\quad \sigma \geq \sigma_Y$		Widely used for Aluminium and Magnesium alloys n =1, Bilinear stress strain law. It is also regarded as one of the parameter, this become a useful analytical expression.

14.	Ramberg and Osgood	$e^* = \sigma^* + K(\sigma^*)^n$ $e^* = \frac{e}{e_o}$, $\sigma^* = \frac{\sigma}{\sigma_o}$ $K = 3/7$	$\log(e^* - \sigma^*)$ Vs $\log \sigma^*$	Particular case of (13), if y=O practicable & applicable to wide variety of materials.
15.	Rao & Legget	$e = \sigma + \beta(\cosh \alpha)\sigma - 1)$		Widely applicable but parameters not easily determinable.
16.	Holmquist & Nadai	$e = \alpha e + \beta \left[e^{(1-\alpha)\frac{\sigma}{\beta}} - 1 \right]$		Parameters difficult to obtain otherwise useful for large strains
O	Osgood	$\sigma = \frac{e(1+be)}{1+ae}$	$(1/\sigma) - (1/e)$ Vs e/σ	For b=O, **reduces** to (8), useful for large strains.
18.	Smith & Sidebottom (26)	$e^* = \sinh S^*$ $e^* = e/e_o$ $S^* = S/S_o$		Two parameters S_o & e_o which are easily determinable for a good fit. Not being used to the extent useful. Widely applicable for **Aluminium** and Steel

2.3.1 Remarks

Out of the so many analytical expressions proposed, only few are suggested as useful. For large strains.Prager's, Holmquists and Nadai's & Osgood's expressions are more suitable than Poncelet's,Ramberg & Osgood's, Rao & Legget's and Smith & Sidebottom expressions mainly because most of these become asymptotic for large strains as is actually observed in practice. First preference however should be given to Ramberg & Osgood's, Smith & Sidebottoms formulas for small strains & Osgood's expression for large strain because of their simplicity & coverage of wide variety of nonlinear structural materials.

2.4 Ramberg-Osgood Function

Ramberg-Osgood[28] originally presented the function in the nondimensional form as

$$\frac{\epsilon}{\epsilon_o} = \frac{\sigma}{\sigma_o} + \frac{3}{7} \left(\frac{\sigma}{\sigma_o} \right)^n \qquad (2.3)$$

For the following relation, also known as Ramberg-Osgood relation,

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B} \right)^n \qquad (2.4)$$

Stress strain constants for various materials are given in Table 3[29].

Table 3

Typical Stress-strain constant (Approximate) for few Materials.

	$E \times 10^{-6}$ psi	B psi	n
Alloy Steel UTS= 100,100 psi; Uniform elongation = 20%	29	122,400	25
Aluminium alloy 2024 -T3: UTS=65,000 psi; uniform elongation =15%.	10	72,300	10
Aluminium alloy 7075-T6: UTS= 3,000 psi; uniform elongation = 10%	10	101,200	10
Magnesium alloy: UTS=39,000 psi; uniform elongation =15%.	6.5	47,500	10

In the present work, generalisation of Ramberg & Osgood's function has been extensively used. For shear stress strain in nondimensional form this can be written as

$$\gamma^* = \alpha(\tau^*) + (\tau^*)^r \qquad (2.5)$$

where and r are arbitrary constats.

& $\tau^* = \frac{\tau}{\tau_0}$, Dimensionless shear stress

$\gamma^* = \frac{\gamma}{\gamma_0}$, Dimensionless shear strain

γ_0 & τ_0 may assume any value required for approximating the actual curve with the above relation. So in a way there are four parsmeters which provide sufficient freedom to choose the values of these parameters for a good fit.

When $\delta = 1$, this relationship is a Ramberg-Osgood function which gives a spectrum of curves (fig. 1) ranging from elastic to elastoplastic cases for r ranging from unity to infinity. For $r > 1$, it is a non-linear relationship. When $\delta = 0$, this relationship corresponds to a parabola of th degree (fig. 2). By adjusting values of δ, r, τ_o & γ_o, any experimental shear stress strain curve can be approximated using equation. (2.5)

2.5 Shear Stress Strain Curves

Till now little has been done to obtain shear stress strain curve of different structural materials, although this is very important before we take up any torsional analysis of non-linear materials. This is because of the experimental difficulties to create such a state of stress and the measurement of the corresponding strain. Buchanan(9) obtained the shear stress-strain curves from the experimental torque-rotation curve by making use of the following relations.

$$\tau = \frac{Tr}{J} \qquad (2.6)$$

where T = the applied torque

r = the radius of the test specimen

J = the polar moment of inertia of the test specimen.

Shear strain $$\gamma = \frac{\theta r}{L} \qquad (2.7)$$

θ = twist in the member

L = length of the member

The above results are true only for circular cross sections. For other sections, these are approximate.

In the next sub-section, we discuss how to obtain shear stress strain curves from unriaxial (tension) stress strain diagram by making **the** of the **von-Mises** Octahedral shear stress theory(28).

2.6 Shear-stress Diagram from Tensile stress strain diagram

Any non-linear stress strain curve consists of elastic portion & plastic portion with the corresponding strains known as elastic and plastic strain respectively. Therefore total shear strain

$$\gamma_{total} = \gamma_{Elastic} + \gamma_{Plastic}$$

$$= \gamma_E + \gamma_P$$

$$= \frac{\tau}{G} + \frac{\tau}{G_P} \qquad (2.8)$$

where G and G_P are modulas of rigidity for elastic and plastic portion respectively and are given by

$$G = \frac{E}{2(1+\mu)} \qquad (2.8.1)$$

$$G_P = \frac{E_P}{2(1+\mu_P)} \qquad (2.8.2)$$

where subscript P on the constants are for plastic portion.

$\mu_P = \frac{1}{2}$ (29) on the assumption that **plstic sstrain-caused** primarily by shear stresses produces no **volume** changes. Now in order to find a method of determining E_P which is a

function of the state of stress, one requirement is that the method must give results that agree with the tensile stress strain diagram. At any point on the tensile stress strain diagram (for plastic strain), the ratio of $\bar{\sigma}/\epsilon_p$ represents E_p. In order to decide what value of $\bar{\sigma}$ should be taken for finding this ratio, use of the von Mises Octahedral shear stress theory has been made. Since the plastic strain are caused by shear stresses, the criteria is based on shear stresses such that Octahedral shear stress produced by two different state of stresses i.e. pure shear and pure tension, be equal. By this assumption, we obtain

$$\bar{\sigma} = \sqrt{3}\,\tau \qquad (2.9)$$

where $\bar{\sigma}$ is the effective normal (tensile) stress fulfilling the above criteria.

So in the process of determining E_p, not only the value of $\bar{\sigma}$ is obtained but at the same time a relation between shear stress & normal tensile stress is also obtained. This relation proves to be of great value in getting the shear stress strain diagram from tensile stress strain diagram.

Substituting (2.9), (2.8.1), (2.8.2) in (2.8) we get,

$$\gamma_{total} = \frac{0.577\,\bar{\sigma}}{G} + \frac{0.577\,\bar{\sigma}}{G_p}$$

$$= \frac{0.577\,\bar{\sigma}}{G} + 1.732\,\epsilon_p \qquad (2.10)$$

Thus knowing G and plastic strain & corresponding stress for pure tension, the shear stress strain curves can be obtained by making use of the above relation. Shear stress-strain diagram have been obtained this way[29,30] (fig. 3-6) for few of the structural materials to show the practical utility of the above method of conversion. Similarly, instead of Octahedral shear stress, we could have used the Maximum shear stress to be equal for the two state of stress mentioned above. But the former is preferred.

2.6.1 Remarks

The use of such a conversion is open to criticism since by doing so we are actually making the maximum shear stresses on Octahedral planes due to two different state of stress as equal; which otherwise may be different on other planes. Nevertheless, in the absence of any shear stress strain diagram, this procedure of conversion is worth doing.

Recently[12] a new method of obtaining shearing strength of concrete has been developed. The setting of testing device and the form of specimen is such that these avoid other secondary failures and give results for pure shear only. This testing device may on further modification be helpful for obtaining shear stress-strain curves of not only concrete but other structural materials.

3. THEORETICAL ANALYSIS

3.1 Introduction

In this chapter the torsion problem for prismatic members of uniform section made up of isotropic and homogenous material has been formulated. The stress strain relation is assumed to be represented by a monotonically increasing continuous function 'f'. It is shown that in general the governing partial differential equation with the boundary conditions is non-linear in nature and hence difficult to solve in closed form. Therefore recourse is taken to Rayleigh - Ritz approach of assuming a function satisfying the boundary conditions such that the complementary energy is minimum. Thus the problem reduces to a minimization problem which is easier to tackle.

3.2 Analysis

Geometry of Deformation

Consider a prismatic bar of arbitrary cross-section of length L, made up of isotropic, homogeneous, inelastic material, subjected to a torque T at the ends as shown in Fig. 7. The torsion member is assumed to be long enough so that method of applying the torque at the ends does not affect the general behaviour at a section away from the ends.

The Z axis is taken along the lenght of the bar

and X, Y axes are taken in the cross-section. The deformations of the twisted bar consist of both the rotation and warping of the cross-section as assumed by Saint - Venant, and may be written as

$$
\begin{aligned}
u &= -\theta z y, \\
v &= \theta z x, \qquad\qquad (3.1) \\
w &= \theta \psi(x,y)
\end{aligned}
$$

where u, v and w are the displacements in X, Y and Z directions repectively, θ is the twist per unit length, and ψ is the warping function.

The non-vanishing strain components are given by

$$
\begin{aligned}
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta \left(\frac{\partial \psi}{\partial x} - y \right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad (3.2) \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z} = \theta \left(\frac{\partial \psi}{\partial y} + x \right)
\end{aligned}
$$

assuming small deformations. By eliminating ψ in equations (3.2), the compatibility equation becomes

$$\frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} = -2\theta \qquad (3.3)$$

Equilibirium Equations

The equilibirium equation for this problem, in the absence of body forces, are

$$
\begin{aligned}
\frac{\partial \tau_{xz}}{\partial z} &= 0 \\
\frac{\partial \tau_{yz}}{\partial z} &= 0 \qquad\qquad (3.4) \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= 0
\end{aligned}
$$

Equations (3.4) express a necessary and sufficient condition for the existence of a stress function $\phi(x,y)$ such that

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = \phi_y$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = \phi_x \qquad (3.5)$$

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$$

where ϕ is an unknown function.

Boundary Conditions

For the lateral surface of the member which is free from external forces, the only boundary condition is

$$\tau_{xz}\, l + \tau_{yz}\, m = 0 \qquad (3.6)$$

where l and m denote the directions cosines of the angles between the outward normal and X and Y axes respectvely.

$$l = dy/ds \quad , \quad m = - dx/ds$$

where s is the arc parameter.

Substituting (3.5) in (3.6), we get

$$\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{d\phi}{ds} = 0 \qquad (3.7)$$

This is satisfied if ϕ is constant on the boundary. Since the stresses are given by the partial derivatives of ϕ , the boundary condition may be written as

$$\phi = 0 \quad \text{on the boundary} \qquad (3.8)$$

At the ends of the bar due to the assumed stress distribution, resultant forces in X and Y direction are

zero. However the couple formed by these forces is equal to the applied torque there and is given by

$$\begin{aligned} T &= \iint_{\text{Area}} (x\,\tau_{yz} - y\,\tau_{zx})\, dxdy \\ &= -\iint_{A} \frac{\partial \phi}{\partial x}\, x\, dx\, dy - \iint_{A} \frac{\partial \phi}{\partial y}\, y\, dx\, dy \\ &= 2\iint_{A} \phi\, dx\, dy \end{aligned} \tag{3.9}$$

Constitutive Equations

The relation between the resultant shear-stress τ and resultant shear strain γ can be writteen in the following general non-dimensional form

$$\gamma^* = f(\tau^*)$$

$$\gamma^* = \frac{\gamma}{\gamma_o}, \quad \tau^* = \frac{\tau}{\tau_o} \tag{3.10}$$

where $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$

and γ_o and τ_o may assume any value required for curve fitting and non-dimensionlization. The function `f' is a single valued monotonically increasing continuous function, which may be linear or non-linear.

In order to express the inelastic stress strain relationship, we make use of Hencky's total strain theory on the following assumptions,

(i) Principle axes of stress and strain coincide.

(ii) Mohr's circle diagrams of stress and strain are similar at any stage in the inelastic deformation,

we get the following relation,

$$\frac{\tau_{xz}}{\gamma_{xz}} = \frac{\tau_{yz}}{\gamma_{yz}} \tag{3.11}$$

Substituting equation (3.11) in (3.10), we obtain

$$\begin{aligned} \gamma_{xz} &= \gamma_o \tau_o \phi_y \; f(\tau^*) \tau^* \\ \gamma_{yz} &= -\gamma_o \tau_o \phi_x \; f(\tau^*) \tau^* \end{aligned} \tag{3.12}$$

Equation (3.12) when substituted in equation (3.3), leads to a non-linear partial differential equation for the stress function ϕ , with the boundary condition that ϕ should be zero along the boundary. However, this equation becomes too complex for a general function 'f'. Hence recourse is taken to the Rayleigh Ritz method.

3.3 Rayleigh Ritz Method

By making use of the principle of minimum complementary energy which holds good even for non-linear materials, this method becomes a powerful tool for solving boundary value problem in the field of elasticity especially when the governing differential equations are difficult to solve. Here a function in the series form with some unknown coefficients A_i ($i = 1, 2, \ldots, n$) is assumed such that boundary conditions are satisfied. These unknown coefficients are determined by minimizing the complementary potential with respect to each A_i, leading to n equations in n unknowns. By solving these, ϕ is completely known and

thus for a given value of ϕ, the torque and the stresses can be calculated using equations (3.9) and (3.5).

3.3.1 Complementary Potential

The complementary potential Π is given by

$$\Pi = U_c - W_e \tag{3.13}$$

where U_c = Total complementary energy in the mumber

$$= \int_{Vol.} U_{co} \, d\,(Vol.) \tag{3.14}$$

where U_{co} = Complementary energy per unit volume.

$$= \int_0^{\tau^*} \gamma_o \tau_o \quad f\,(\tau^*)\; d\tau^* \tag{3.15}$$

and W_e = the total complementary work done by the external torque acting through the angle of twist θ.

$$= 2\,L \iint_A \theta\,\phi\; dx\,dy \tag{3.16}$$

Substituting equations (3.14), (3.15) and (3.16) in (3.13), we get,

$$\Pi = L\,\gamma_o \tau_o \int \iint_A \left\{ \int_0^{\tau^*} f(\tau^*)\; d\tau^* - \frac{2\theta\phi}{\gamma_o \tau_o} \right\} dx\,dy \tag{3.17}$$

4. APPLICATIONS

4.1 Introduction

In order to demonstrate the applicability of the theoretical analysis discussed in previous chapter, two cases, rectangular and circular cross-sections are considered in this chapter. Smilar procedure can be extended to other practical cross-sections. From stress strain relation stand point, only generalised Ramberg-Osgood function is replaced for the general function 'f'. This is done with the view that such a function with sufficient member of parameters covers almost all non-linear materials and hence the present analysis can be applied to a wide variety of materials. Although here also, it can be extended for some other stress strain function 'f' which may be more suited to any particular material.

4.2 Idealised Non-linear Stress Strain Law and Complementary Potential

Generalised Ramberg-Osgood function can be applied to a general class of non-linear materials and is given by

$$\gamma^* = \delta\tau^* + (\tau^*)^r \tag{4.1}$$

For this relationship the complementary potential π can be written as

$$\pi = L\,\gamma_o\tau_o \iint_A \left\{ \frac{\delta}{2}\left[\left(\frac{\phi_x}{\tau_o}\right)^2 + \left(\frac{\phi_y}{\tau_o}\right)^2\right] + \frac{1}{2m}\left[\left(\frac{\phi_x}{\tau_o}\right)^2 + \left(\frac{\phi_y}{\tau_o}\right)^2\right]^m - \frac{2\,\theta\phi}{\gamma_o\tau_o}\right\} dx\,dy \tag{4.2}$$

where $m = \frac{r+1}{2}$

Another curve of interest is of the form

$$\gamma^* = \sinh \tau^* \tag{4.3}$$

Which has been extensively studied by Smith and Sidebottom (11) for the study of circular, rectangular, and triangular cross-section with particular reference to steel members. This curve has been appoximated by the Ramberg - Osgood function by taking the first two terms and the results have been discussed in section 7.

4.3 Circular Cross-Section

For a circular cross-section of radius 'a' and length L, the function ϕ can be assumed in general as follows,

$$\frac{\phi}{\tau_o}(x,y) = (\sum_{i=0}^{p} \sum_{j=0}^{q} A_{ij}\, x^i\, y^j\ (x^2+y^2-a^2)$$

We retain only the terms with coefficient A_{oo}, A_{20}, A_{o2}. Because of symmetry $A_{20} = A_{02}$

$$\frac{\phi}{\tau_o}(x,y) = \left\{A_{oo}+A_{20}\ (x^2+y^2)\right\}\ (x^2+y^2-a^2) \tag{4.4}$$

or in polar co-ordinates

$$\frac{\phi}{\tau_o}(r,\theta) = (A+Br^2)\ (r^2-a^2) \tag{4.5}$$

where $A = A_{oo}$

$B = A_{20}$

$$(\frac{\phi_x}{\tau_o})^2+(\frac{\phi_y}{\tau_o})^2 = 4\ R^2 \left[A_1+2B_1\ R^2-B_1\right]^2 \tag{4.6}$$

where $R = \frac{r}{a}$

$A_1 = A\,a$

$B_1 = B\,a^3$

Substituting (4.5), (4.6) in (4.2) and integrating we obtain

$$\frac{\pi}{L\,\gamma_o \tau_o} = \delta \pi a^2 \left[A_1^2 + \frac{1}{3} B_1^2 + \frac{2}{3} A_1 B_1\right] + \frac{4^m \pi a^2}{m} \int_0^1 R^{2m+1} \left[A_1 + 2 B_1 R^2 - B_1\right]^{2m} dR + \frac{\pi}{2} \frac{\theta h}{\gamma_o} a^2 \left(A_1 + B_1/3\right) \qquad (4.7)$$

where $h = 2a$

To minimize this complementary potential in order to determine the value of the unknown constants A and B, we equate the partial derivatives of the complementary potential w.r.t. A and B each equal to zero

$$\frac{\partial \pi}{\partial A} = 0$$

or $$\delta\left[A_1 + \frac{1}{3} B_1\right] + 4^m \int_0^1 R^{2m+1} (A_1 + 2B_1R^2 - B_1)^{2m-1} dR + \frac{1}{4} \frac{\theta h}{\gamma_o} = 0 \qquad (4.8)$$

and $$\frac{\partial \pi}{\partial B} = 0$$

or $$\frac{\delta}{3}\left[A_1 + B_1\right] + 4^m \int_0^1 R^{2m+1} (A_1 + 2B_1R^2 - B_1)^{2m-1} (2R^2 - 1)\, dR + \frac{1}{12} \frac{\theta h}{\gamma_o} \qquad (4.9)$$

These two simultaneous nonlinear integral equations are to be solved numerically for A_1 and B_1 by the method described in the next chapter.

For a given value of θ, the torque T is given by

$$\frac{T}{\tau_o \ \pi a^2 \ 2a} = \tfrac{1}{2}\left[A_1 + B_1/3\right] \qquad (4.10)$$

and maximum is given by

$$\tau^*_{max} = 2\ (A_1+B_1) \qquad (4.11)$$

4.4 Rectangular Cross-section

For the case of rectangular Cross-section, of width 2a, depth 2b and length L, stress function ϕ can be assumed in general form as

$$\frac{\phi(x,y)}{\tau_o} = \sum_{i=0}^{p} \sum_{j=0}^{q} A_{ij}\, x^i y^j \quad (x^2-a^2)(y^2-b^2) \qquad (4.12.1)$$

or

$$= \sum_{i=1,3.}^{p} \sum_{j=1,3..}^{q} A_{ij} \operatorname{Cos} \frac{i\pi x}{2a} \operatorname{Cos} \frac{j\pi y}{2b} \qquad (4.12.2)$$

Retaining only the terms with coefficient A_{oo}, A_{o2} and A_{20} in equation (4.12.1), we get

$$\frac{\phi}{\tau_o} = (A_{oo}+A_{20}\ x^2+A_o\text{-}y^2)(x^2-a^2)(y^2-b^2)$$

$$= (A+Bx^2+Cy^2)(x^2-a^2)(y^2-b^2) \qquad (4.13)$$

$$\left(\frac{\phi_x}{\tau_o}\right)+\left(\frac{\phi_y}{\tau_o}\right) = 4K^2\Big[K^2X^2(Y^2-1)^2\left\{A_1+B_1(2X^2-1)+K^2C_1Y^2\right\}^2 + Y^2(X^2-1)^2\left\{A_1+B_1X^2+C_1K^2(2Y^2-1)\right\}^2\Big] \quad (4.14)$$

where $K = b/a$

$X = x/a$

$Y = y/b$

Substituting (4.13) and (4.14) in (4.2), we get

$$\frac{\pi}{L\tau_o\gamma_o} = \frac{a^2\delta}{2}\Big[2.84444(K^2+1)\;K^3A_1^2+(0.893968\;K^2 + 0.135450)\;K^3\;B_1^2 +(0.893968+0.812699\;K^2)\;K^7C_1^2 + (1.13778\;K^2+0.812699)\;K^3A_1B_1 + (1.13778+0.812699\;K^2)\;K^5A_1C_1+0.162540\;(K^2+1)\;K^5B_1C_1\Big]+\frac{2ab}{m}\int_0^1\!\!\int_0^1\left[\left(\frac{\phi_x}{\tau_o}\right)^2+\left(\frac{\phi_y}{\tau_o}\right)^2\right]^m dx\,dy - \frac{\theta h}{\gamma_o}\left\{\frac{16}{45}\,a^2K^3\,(5A_1+B_1+K^2\,C_1)\right\} \quad (4.15)$$

where $A_1 = Aa^3$

$B_1 = Ba^5$

$C_1 = Ca^5$

$h = 2a$

To determine A, B and C, we equate the partial derivatives of (4.14) w.r.t. A, B and C equal to zero. Thus we get

$$\frac{\partial \pi}{\partial A} = 0$$

$$\text{or} \quad \frac{\delta}{32}\Big[5.68888\,(K^2+1)\,A_1 + (1.13778\,K^2+0.812699)\,B_1 + K^2(1.13778 + 0.812699\,K^2)\,C_1\Big]$$
$$+ 4^{m-1}\,K^{2m-2}\int_0^1\!\!\int_0^1 \Big[K^2X^2(Y^2-1)\,\{A_1+B_1\,(2X^2-1) + C_1Y^2\}^2 + Y^2(X^2-1)^2\{A_1+B_1X^2+C_1(2Y^2-1)\}^2\Big]^{m-1}$$
$$\Big[K^2X^2\,(Y^2-1)^2\{A_1+B_1(2X^2-1)+K^2C_1Y^2\} + Y^2(X^2-1)^2\{A_1+B_1X^2+K^2C_1(2Y^2-1)\}\Big]\,dX\,dY$$
$$- \frac{\theta h}{9\gamma_o} = 0 \qquad (4.16)$$

$$\frac{\partial \pi}{\partial B} = 0$$

$$\text{or} \quad \frac{\delta}{32}\Big[(1.1787936\,K^2+0.270900)\,B_1+(1.13778\,K^2+ + 0.812699)\,A_1 + K^2\{0.162540\,(K^2+1)\,C_1\}\Big]$$
$$+ 4^{m-1}\,K^{2m-2}\int_0^1\!\!\int_0^1\Big[K^2X^2(Y^2-1)^2\{A_1+B_1(2X^2-1) + K^2C_1Y^2\}^2 + Y^2(X^2-1)^2\{A_1+B_1X^2+K^2C_1(2Y^2-1)\}^2\Big]^{m-1}.$$
$$\Big[K^2X^2(Y^2-1)^2\{A_1+B_1(2X^2-1) + K^2C_1Y^2\}(2X^2-1) + Y^2\,(X^2-1)^2\{A_1+B_1X^2 + K^2C_1\,(2Y^2-1)\}\,X^2\Big]\,dX\,dY$$

$$- \frac{\theta h}{45\gamma_o} = 0 \qquad (4.17)$$

$$\text{and} \quad \frac{\partial \pi}{\partial C} = 0$$

or

$$\frac{\delta}{32}\Big[(1.1787936+0.270900\,K^2)\,K^2C_1+(1.13778+0.812699\,K^2)\,A_1+0.162540\,(K^2+1)\,B_1\Big]$$

$$+\,4^{m-1}\,K^{2m-2}\int_0^1\!\!\int_0^1\Big[K^2X^2(Y^2-1)^2\{A_1+B_1(2X^2-1)+K^2C_1Y^2\}^2+Y^2(X^2-1)^2\{A_1+B_1X^2+K^2C_1(2Y^2-1)\}^2\Big]^{m-1}\Big[K^2X^2(Y^2-1)^2\{A_1+B_1(2X^2-1)+K^2C_1Y^2\}Y^2+Y^2(X^2-1)^2\{A_1+B_1X^2+K^2C_1(2Y^2-1)\}(2Y^2-1)\Big]\,dX\,dY-\frac{\theta h}{45\,Y_o}$$

(4.18)

Again these three non-linear equations (4.16), (4.17) and (4.18) are to be solved numerically for A_1, B_1 and C_1 by the method given in the next chapter.

For a given θ, torque T is given by

$$\frac{T}{\tau_o\;4ab\;\;2a}=\frac{4}{45}\;K^2\;(A_1+B_1+C_1) \qquad (4.19)$$

and maximum τ is given as

$$\tau^*_{max}\;=\;2\;K^2\;(A_1+B_1) \qquad (4.20)$$

5. NUMERICAL TECHNIQUES USED

5.1 Introduction

Problem encountered in the previous chapter can be attempted either analytically or numerically. Analytical solution, however, does not seem possible here, firstly, because the simultaneous equations are of non-linear in nature and secondly the integral involved in each of the equations can not be evaluated for non-integral values of 'm' in general, unless the values of A,B, C are known. Therefore a numerical technique is suggested and used for the solution of the above problem. A particular numerical procedure is the usual Newton's Iterative technique to solve the system of non-linear equations. At every iteration first the integral involved in each of the equations is evaluated by the Simprous Rule, for the values of A,B,C obtained at previous iteration. To start with the method, suitable values of A,B,C are chosen. Once the integrals are evaluated, the system of nonlinear integral equations reduces to system of non-linear equations and therefore can be solved by Newton's method.

5.2 Simpson's Rule for Numerical Integration

For definite single integration, the simpson's rule

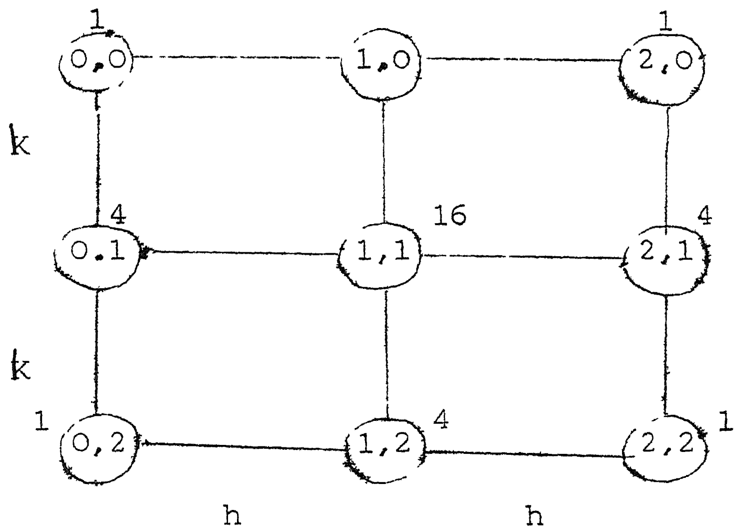
$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+\ldots..\right) \qquad (5.1)$$

where y_o, y_1 .... denote the ordinates at $x_o, x,$ ...... etc and h is interval length.

For double integration, the extension of the above rule known as Extended Simpson's Rule is as follows,

$$\int_{x_o}^{x_o+2h}\int_{y_o}^{y_o+2K} Z\,dy\,dx = \frac{hK}{9}\left[Z_{oo}+Z_{o2}+Z_{22}+Z_{20}+4(Z_{01}+Z_{12}+Z_{21}Z_{10})+16Z_{11}\right] \qquad (5.2)$$

where Z_{oo}, Z_{o1} ...., denote the functional values at grid points and h, k are grid lengths



The coeff. of Z^s are shown above in the figure By adding any number of such unit blocks the whole domain of integration can be covered.

5.3 Solution of Simultaneous Non-linear Equations (Nowton's Method)[31]

Let the three equations be

$$
\begin{aligned}
f(x,y,z) &= 0 \\
g(x,y,z) &= 0 \qquad\qquad (5.3) \\
h(x,y,z) &= 0
\end{aligned}
$$

Let (x_0, y_0, z_0) be an approximation to the solution (ξ, η, ζ) of equation (5.3).

Assuming that f,g and h are differentiable, we expand $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ about (x_0, y_0, z_0), by Taylor's Series for functions of three variables as

$$
\begin{aligned}
f(x,y,z) = {} & f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) \\
& + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0) \\
& (z-z_0) + \ldots\ldots\ldots \\
g(x,y,z) = {} & g(x_0,y_0,z_0) + g_x(x_0,y_0,z_0)(x-x_0) \\
& + g_y(x_0,y_0,z_0)(y-y_0) + g_z(x_0,y_0,z_0) \\
& (z-z_0) + \ldots\ldots\ldots
\end{aligned}
\qquad (5.4)
$$

$$
\begin{aligned}
h(x,y,z) = {} & h(x_0,y_0,z_0) + h_x(x_0,y_0,z_0)(x-x_0) \\
& + h_y(x_0,y_0,z_0)(y-y_0) + h_z(x_0,y_0,z_0) \\
& (z-z_0) + \ldots\ldots
\end{aligned}
$$

Further, assuming that (x_0, y_0, z_0) is sufficiently close to the solution (ξ, η, ζ), higher order terms can be neglected. Therefore equating the linear terms to zero, we obtain

$$
\begin{aligned}
f_x\,(x-x_o)+f_y(y-y_o)+f_z(z-z_o) &= -\,f \\
g_x\,(x-x_o)+g_y(y-y_o)+g_z(z-z_o) &= -\,g \qquad (5.5)\\
h_x\,(x-x_o)+h_y(y-y_o)+h_z(z-z_o) &= -\,h
\end{aligned}
$$

where it is understood that all functions and their partial derivative are evaluated at (x_o,y_o,z_o). We then expect that the new solution (x_1,y_1,z_1) of (5.5) will be closer to the solution (ξ,η,ζ) than (x_o,y_o,z_o). Then by Cramer's rule

$$
x_1-x_o = \frac{\begin{vmatrix} -f & f_y & f_z \\ -g & g_y & g_z \\ -h & h_y & h_z \end{vmatrix}}{\begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}} = \frac{\begin{vmatrix} -f & f_y & f_z \\ -g & g_y & g_z \\ -h & h_y & h_z \end{vmatrix}}{J\,(f,g,h)}
$$

$$
y_1-y_o = \frac{\begin{vmatrix} f_x & -f & f_z \\ g_x & -g & g_z \\ h_x & -h & h_z \end{vmatrix}}{J(f,g,h)}
$$

$$
z-z_o = \frac{\begin{vmatrix} f_x & f_y & -f \\ g_x & g_y & -g \\ h_x & h_y & -h \end{vmatrix}}{J(f,g,h)}
$$

Provided $J(f,g,h)$ at $(x_o, y_o, z_o) \neq 0$

The solution (x_1, y_1, z_1) of this system now provides a new approximation to (ξ, η, ζ)

In this way we generate successive approximation till the convergence is achieved,

$$Xx_{i+1} = x_i + \left[\frac{\begin{vmatrix} -f & f_y & f_z \\ -g & f_y & g_z \\ -h & h_y & h_z \end{vmatrix}}{J(f,g,h)} \right]_i$$

$$y_{i+1} = y_i + \left[\frac{\begin{vmatrix} f_x & -f & f_z \\ g_x & -g & g_z \\ h_z & -h & h_z \end{vmatrix}}{J(f,g,h)} \right]_i$$

$$z_{i+1} = z_i + \left[\frac{\begin{vmatrix} f_x & f_y & -f \\ g_x & g_y & -g \\ h_x & h_y & -h \end{vmatrix}}{J(f,g,h)} \right]_i$$

5.3.1 Convergence of Newton's Method

The above method when converges, it converges quadratically. It has been shown [31] that this method will converge under the following sufficient conditions;

(1) f,g,h and all their derivative through second order are continuous and bounded in a region R containing (ξ, η, ζ)

(2) The Jacobian J (f,g,h) does not vanish in R.

(3) The initial approximation is sufficiently close to the root (ξ, η, ζ)

5.4 Selection of Sub-intervals and Convergence Limit

For stable and sufficiently accurate results, it is necessary to select suitable no. of sub-intervals for the numerical integration. Although efficient computation demands the no. of intervals to be as small as possible, it is rather difficult to put analytically any limit on it to ensure stable and accurate results. These are best selected by means of trials. For the present problem, twenty subintervals have been found to be adequate.

Convergence limit which control the no. of iterations are selected depending upon how the results are going to be used. For plotting purposes twenty iterations or convergence limit 10^{-6} whichever is reached earlier is quite sufficient.

6. ULTIMATE STRENGTH OF PLAIN CONCRETE IN TORSION

6.1 Introduction

For the design of concrete structures prediction of ultimate strength is as important as the behaviour from begining to failure. In this section a methodology has been presente for obtaining the ultimate strength of plain concrete in torsion from uniaxial test results. Sinh curve has been used to represent the shear stress strain relation. Result for rectangular cross section have been obtained and compared with other theories and experimental results. Such a methodology can be extended to other sections and stress-strain, functions.

6.2 Tensile Strength of Concrete

The shape of the stress strain curve in pure shear ($\tau - \gamma$) is similar to the shape of the stress-strain curve in uniaxial tension ($\sigma - \epsilon$)[29]. The conversion factors depend upon the failure theory applicable for the material. For example, the octahodral shear-stress theory which is applicable to steel, gives[29].

$$\tau = \frac{\sigma}{\sqrt{3}} \quad \text{and} \quad \gamma = \sqrt{3}\,\epsilon \qquad (6.1)$$

For concrete, the above expressions are very conservative Mohr's failure theory is applicable for converting the maximum shear stress to equivalent uniaxial tensile failure stress. This can be written as

$\bar{\tau} = \alpha f_t'$ where f_t' = ultimate tensile stress of the material

The value of α using a straight lime simplification[13] of Mohr's theory is $\frac{f'_c}{f'_c + f'_t}$, while Iyengar[33] recommend an approximate value of $\frac{2f'_c - f'_t}{2f'_c}$, where f'_c = ultimate cylindrical strength of concrete. McHenry and Karni[34] also give values of from their experimental tests. α varies from 0.85 to 0.95. The value of f'_t for design purposes given by Hsu[13] for stone concretes is

$$f'_t = \beta \sqrt{f'_c} \qquad \text{where} \quad \beta = 5.0 \qquad (6.2)$$

However, in analysing test results β may vary between 4.5 to 6.5 . For high strength concretes Navaratnarajah[35] uses the following relationship

$$f'_t = 0.68\ (f'_c)^{\frac{3}{4}} \qquad (6.3)$$

The tensile strain ϵ for a given value of shearing strss can be written as[33]

$$\gamma = 2(1+\mu)\epsilon \qquad (6.4)$$

where μ = poisson's ratio, which is assumed to be constant over the whole range of loading, with a value of 0.17[13]. The normal stress-strain relation $(\sigma - \epsilon)$ from $(\tau - \gamma)$ relationship is therefore given for the Sinh relationship (equation 4.3) as

$$\frac{2 \quad (1+\mu)\epsilon}{\gamma_o} = \text{Sinh}\ \frac{\sigma}{\alpha\tau_o}$$

At ultimate, the value of σ is $\bar{\sigma}$ with a value of $\alpha f'_t$ and the value of ϵ is $\bar{\epsilon}$. The value of $\bar{\epsilon}$ from direct tension stress-strain curves is reported in the litearature to be around 150 x 10^{-6} by Hsu, 200 - 400 x 10^{-6} by Evans and Marathe[36], 400 x 10^{-6} by Mirza[37]. Further research is needs to find the parameters influencing this value.

6.3 Ultimate Strength Methodology

With the above considerations a methodology for finding the ultimate strength, angle of twist per unit length $\bar{\theta}$ of prismatic plain concrete members in pure torsion, is herein presented using the Sinh curve idealisation for behaviour of plain concrete. This is applied to the case of rectangular beams. Now

$$\frac{\gamma}{\gamma_o} = \text{Sinh}\,\frac{\tau}{\tau_o} \qquad (6.5)$$

where γ_o, τ_o are to be determined in terms of f'_c, f'_t $\bar{\epsilon}$, α, β. It can be readily seen that $\frac{\tau_o}{\gamma_o} = G_o$ at the origin, where

$$G_o = \frac{E}{2(1 + \mu)} = \text{shear modulus} \qquad (6.6)$$

Equation 4.3 can be approximated as

$$\bar{\gamma} = \frac{\bar{\tau}}{G_o} + \frac{1}{6G_o}\,\frac{\bar{\tau}^3}{\tau_o^2}$$

by neglecting higher order terms in the Sinh expansion Using the above expressions, the following relation is obtained

$$\tau_o = \sqrt{\frac{\bar{\tau}^3}{6(G_o\bar{\gamma} - \bar{\tau})}}$$

The value of E in tension is taken as the same as in compression after Todd; Oladapo[35] and Hsu[13].

The value of E can be written as $= 57,500 \sqrt{f'_c}$

for stone concretes. Using $\bar{\epsilon} = 250 \times 10^{-6}$ $\alpha = 0.85$

$\beta = 5$ the value of can be written as $\tau_o = \dfrac{\bar{\tau}}{3.8}$.

Example

Using the results of Smith and Sidebottom[10], for O/a of 1.0 the value of

$$\frac{\bar{\theta}\ 2a\sqrt{3}}{\gamma_o} = 54$$ and the ultimate torque $\bar{T}$ is given by

$$\frac{T}{8a^2b} = K_1 f'_t$$

where $K_1 = 0.254$.

The coefficient 0.254 in the above expression is only a probable value and may vary between 0.23 to 0.275. For elastic analysis neglecting the presence of compression stress the coefficient is 0.208 while it is 0.33 by rigid plastic theory.

7. RESULTS

7.1 Summary

Firstly a stress function satisfying the boundary conditions was assumed. It contained two and three unknown parameters for rectangular and circular cross-section respectively. Substituion of stress function in the complementary potential derived for stress strain function given by Ramberg & Osgood, resulted finally in making the complementary potential a function of the unknown parameters. These were determined by making use of the principle of minimum complementary potential. This minimisation w.r.t. each of the unknown parameters led to a system of non-linear integral equation, which were solved numerically on IBM 7044 for different values of twist by the numerical technique described in section 5. The computer program is given in the appendix.

Once the stress function is known, T^* Vs θ^* and τ^*_{max} Vs θ^* curves are obtained making use of equations 4.10, 4.11 for circular cross-Section and equations 4.19, 4.20 for rectangular cross-section. All this done on the assumption that the values of the unknown parameters of the idealised shear stress strain curve are known.

7.2 Circular Cross-section

Results are obtained for different materials. For concrete the value of r = 1.10 to 1.16 for δ = 1 was suggested by Marshall & Tembe, while the parabolic curve

($r = 2$, $\varsigma = 0$) was used by Miyambo, in their analysis of circular cross-section in pure forsion. Curve for T^* Vs θ^* for different values of r and δ for concrete are shown in fig. 8.

For other non-linear materials like Alluminium, Magnesium and Steel, the T^* Vs θ^* curve is shown in fig. 9. The values of parameter of Ramberg & Osgood function for these materials are given in section 2.

7.2.1 Comparison

Results are compared with Buchanan's experimental results for beams C_1, C_2 and C_3. The $\tau - \gamma$ curves obtained from experimental results are approximated by hyperbolic Sinh function and Ramberg-Osgood function. The comparison for T Vs θ and T^* Vs θ^* for these functionsis done in fig. 10 and fig 11 respectively.

Besides this, results are also compared with Smith & Sidebottom. The hyperbolic Sinh stress strain relation used by them is approximated by the Ramberg Osgood function by taking the first two term of the expansion as given below

$$\gamma^* = \sinh \tau^*$$

$$\simeq \tau^* + \frac{1}{3!} (\tau^*)^3$$

$$= \frac{\tau}{\tau_o} + \frac{1}{6} \left(\frac{\tau}{\tau_o}\right)^3$$

$$\gamma/\gamma_o = 6\,(\tau/\tau_o) + (\tau/\tau_o)^3$$

$$\delta = 6$$

$$r = 3 \text{ i.e. } m = 2$$

For these values of γ and δ, results for T^* Vs θ^* and τ^*_{max} Vs θ^* are compared in fig. 12. Results for T^*, τ^* for various values of θ^*, r and δ are given in Tabulalar form in Tables 4 and 5.

6.3 Rectangular Cross-section

For this case curves for T^* Vs θ for different values of r, $\delta = 0$ for concrete are presented for b/a = 1.0 1.5, 2.0 in figures 13,14 and 15 respectively.

T^* Vs θ^* curves for aluminium, magnesium Steel and reinforced plastics are shown in fig. 16.

6.3.1 Comparison.

For rectangular case also, results are compared with Buchanan's experimental results. The τ - rγ curve was idealized by Sinh & Ramberg Osgood functions. The curves for T^* Vs θ^* are compared in fig. 17. Ramberg-Osgood function approximation give better results than Sinh function.

Comparisons with Sidebottom & Smith's analytical results are shown in figs. 18 & 19, for T^* Vs θ^* and τ^*_{max} Vs θ^* respectively. The Sinh function is approximated by taking first two terms of the exparison as given earlier.

Results for T^*, τ^*_{max} for various values of θ^* r, δ and b/a are given in tabular form in Tables 6-9.

Ultimate Strength

The values of K_1 by the proposed method (section 6) are shown in Table 10 and compared with values obtained by elastic and plastic analysis. From Table 10, it can be seen

that the present analysis gives a value of t ultimate strength, about 78% of that by plastic analysis. This is in comformity to the findings of Collins[38] who proposed the Nadai's sand heap analogy with the value of the stress to be $3.5\sqrt{f'_c}$. To check the validity, of above ultimate torque relations with experimental result of rectangular beams, comparison is made in Table 11. The value of f'_t is taken to be the value obtained by direct tests (where available), or the maximum of the value obtained from the expressions given in this section. In the tests reported by Collins,[38] the value of f'_t is taken as 0.56 times the value from split cylinder tests.

From Table 11 it is seen that the ultimate torque obtained agrees very closely, when the actual value of is known and is conservative when $\beta = 5.0$ is used. The value of twist at ultimate is not compared, as this is very much dependent on the value of $\bar{\epsilon}$ assumed in the general expressions.

7.4 Discussion

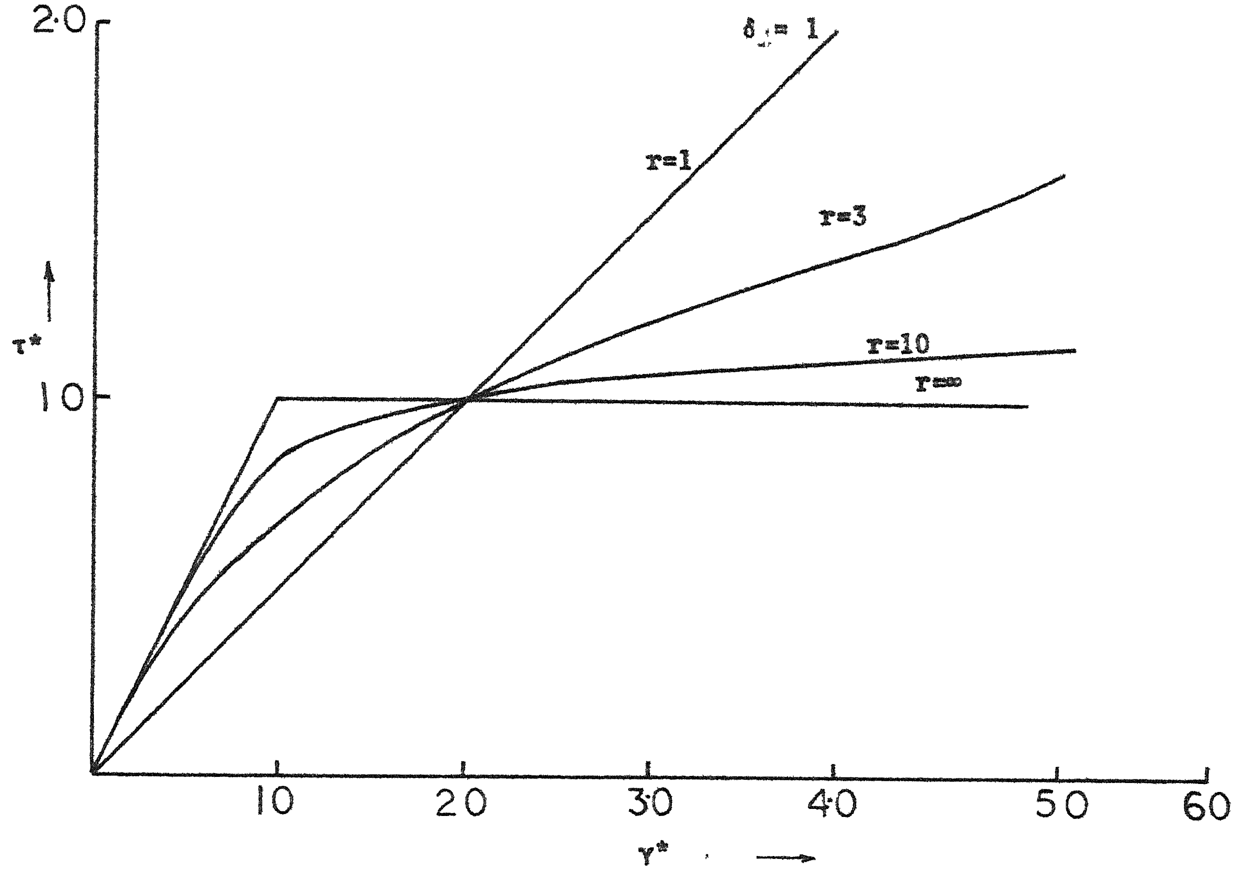
Results have been presented for five structural materials. However, the comparison with the experimental results is done for concrete only since these were the only available. For both cases, circular and rectangular cross-sections, two curves have been used for approximating the given $\tau - \gamma$ curve. In both cases, Ramberg-Osgood function give better results than Sinh function, although both these slightly over estimate the torque. More experimental results for different materials need to be obtained and compared before

a definite conclusion can be made as to which function give better results for a particular material.

Comparison with the analytical results of Smith & Sidebottom, obtained after approximating the Sinh function by Ramberg Osgood function, show good agreement particularly for circular case. A point to be worth noting is that analytical results are exact for circular case as given by Smith and Sidebottom. For rectangular cross-section at one of the places in derivation Cosh ξ is replaced by $= 1 + m\xi^2 + n\xi^4$, $m = 0.47185$ & $n = 0.057442$. This ensures maximum error of 1.5% if $0 < \xi < 3$.

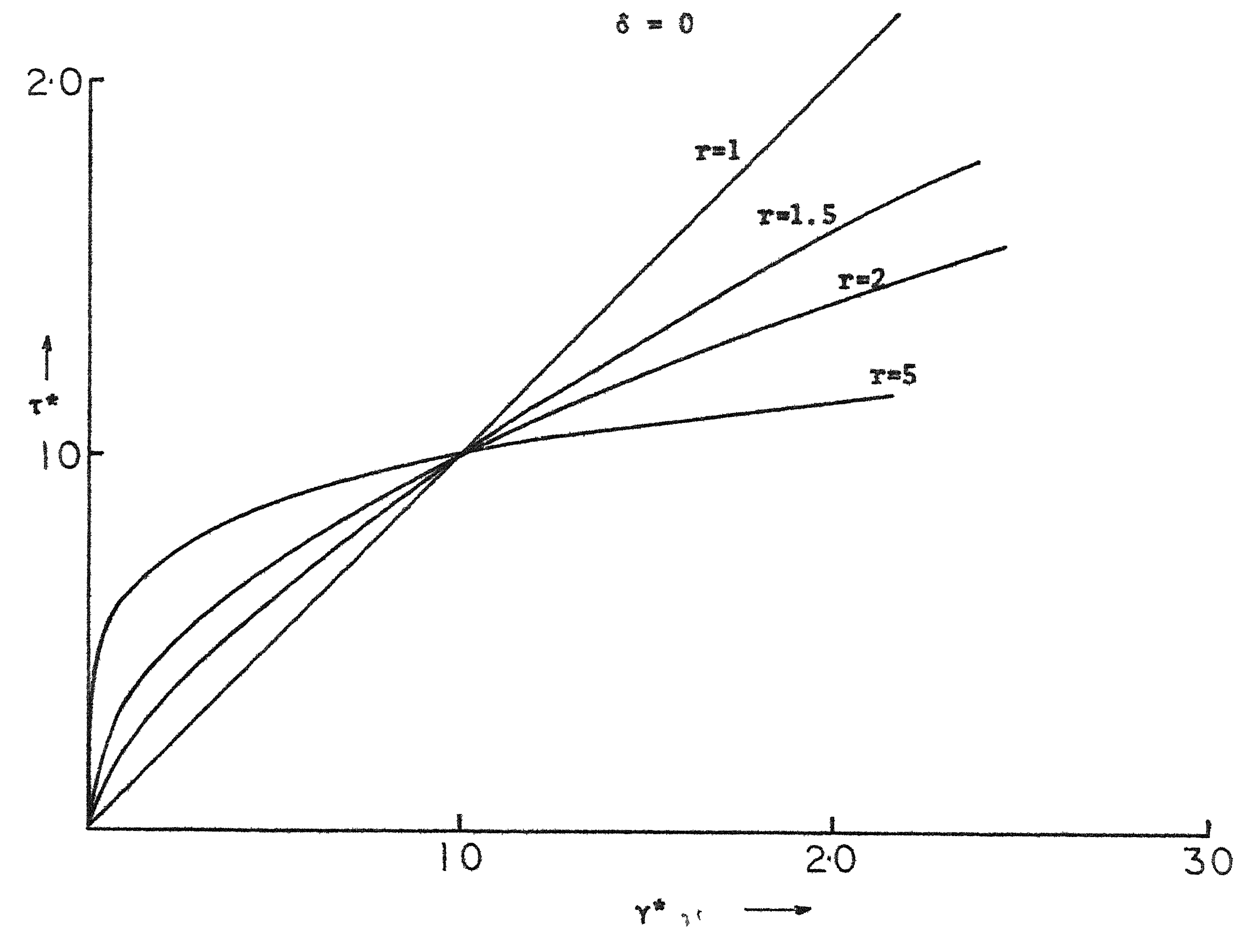
Proposed ultimate strength methodology for plain concrete on comparison with elastic & plastic theories and experimental results show good agreement.

However, more research is needed on the tensile stress-strain relationship for predicting the ultimate twist, as this is very much dependent on the value of ultimate strain.

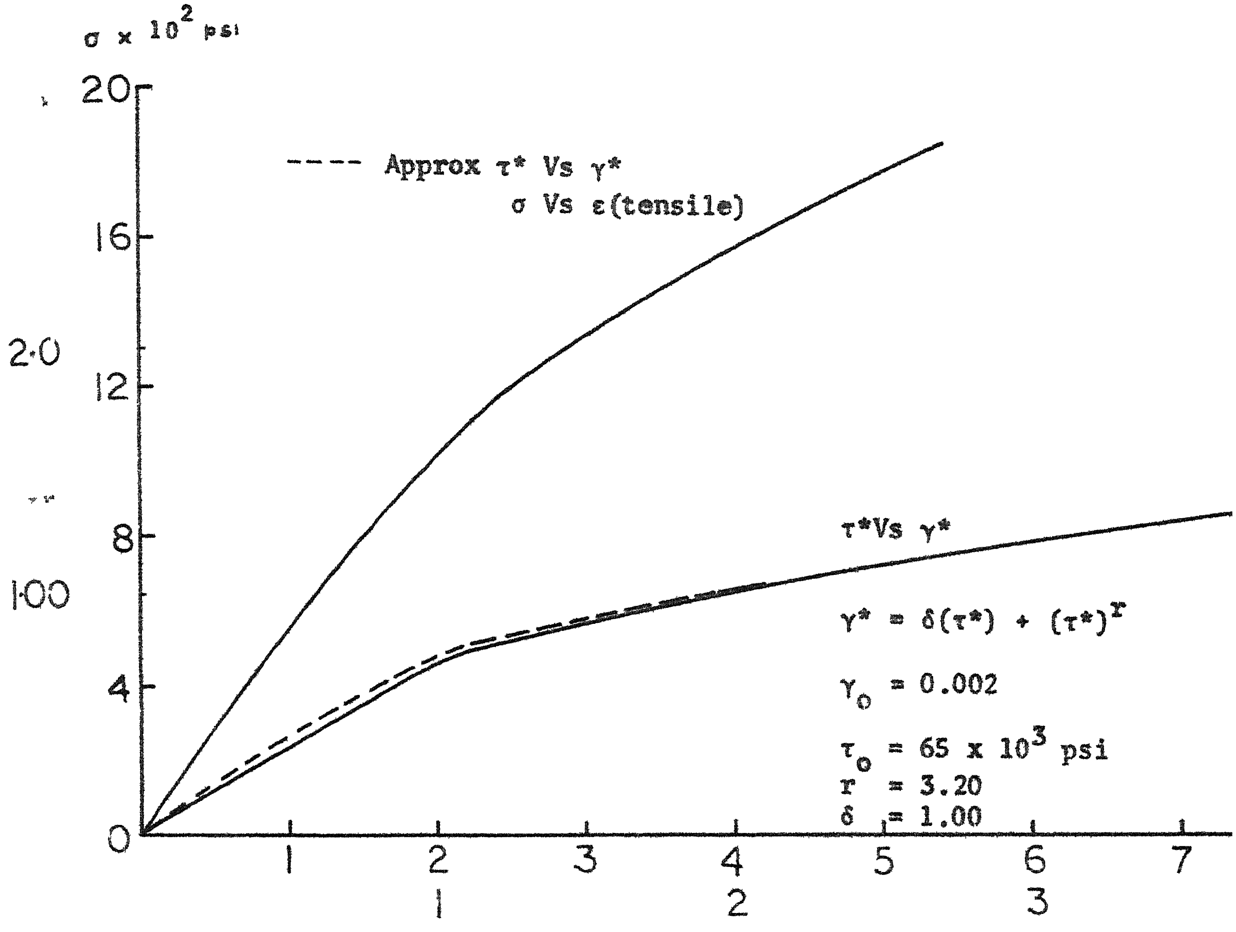


δ = 1
r=1
r=3
r=10
r=∞
τ*
γ*
2·0
1·0
1·0
2·0
3·0
4·0
5·0
6·0

Fig. 1



δ = 0
r=1
r=1.5
r=2
r=5
τ*
γ*
2·0
1·0
1·0
2·0
3·0



σ x 10² psi
20
16
12
8
4
0
2·0
1·00
---- Approx τ* Vs γ*
σ Vs ε(tensile)
τ*Vs γ*
γ* = δ(τ*) + (τ*)^r
γ₀ = 0.002
τ₀ = 65 x 10³ psi
r = 3.20
δ = 1.00
1
2
3
4
5
6
7
1
2
3

Fig. 3

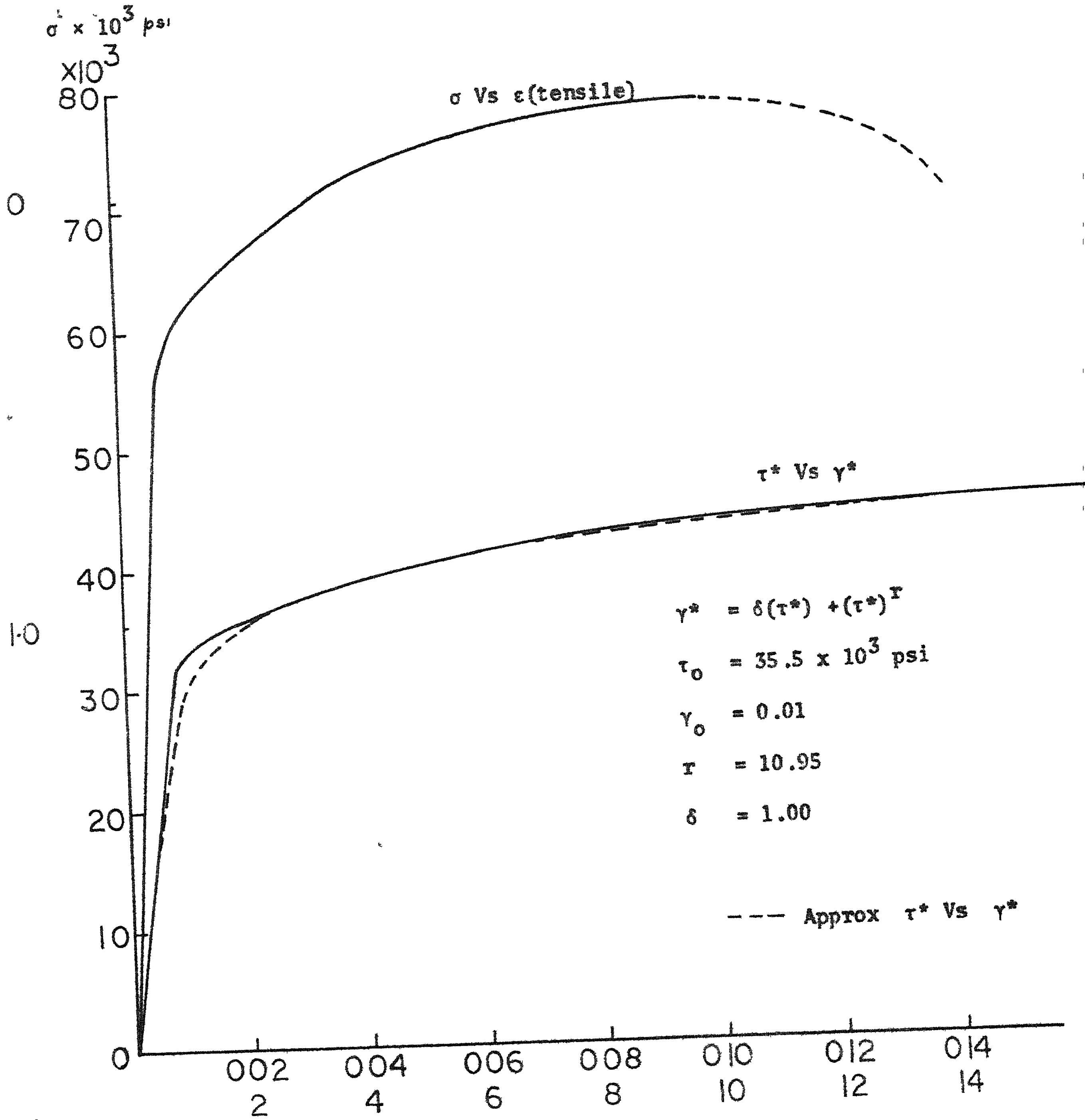

σ × 10³ psi
x10³
σ Vs ε(tensile)
τ* Vs γ*
γ* = δ(τ*) +(τ*)^r
τ₀ = 35.5 x 10³ psi
γ₀ = 0.01
r = 10.95
δ = 1.00
--- Approx τ* Vs γ*
0
10
20
30
40
50
60
70
80
1·0
002
004
006
008
010
012
014
2
4
6
8
10
12
14

Fig. 4

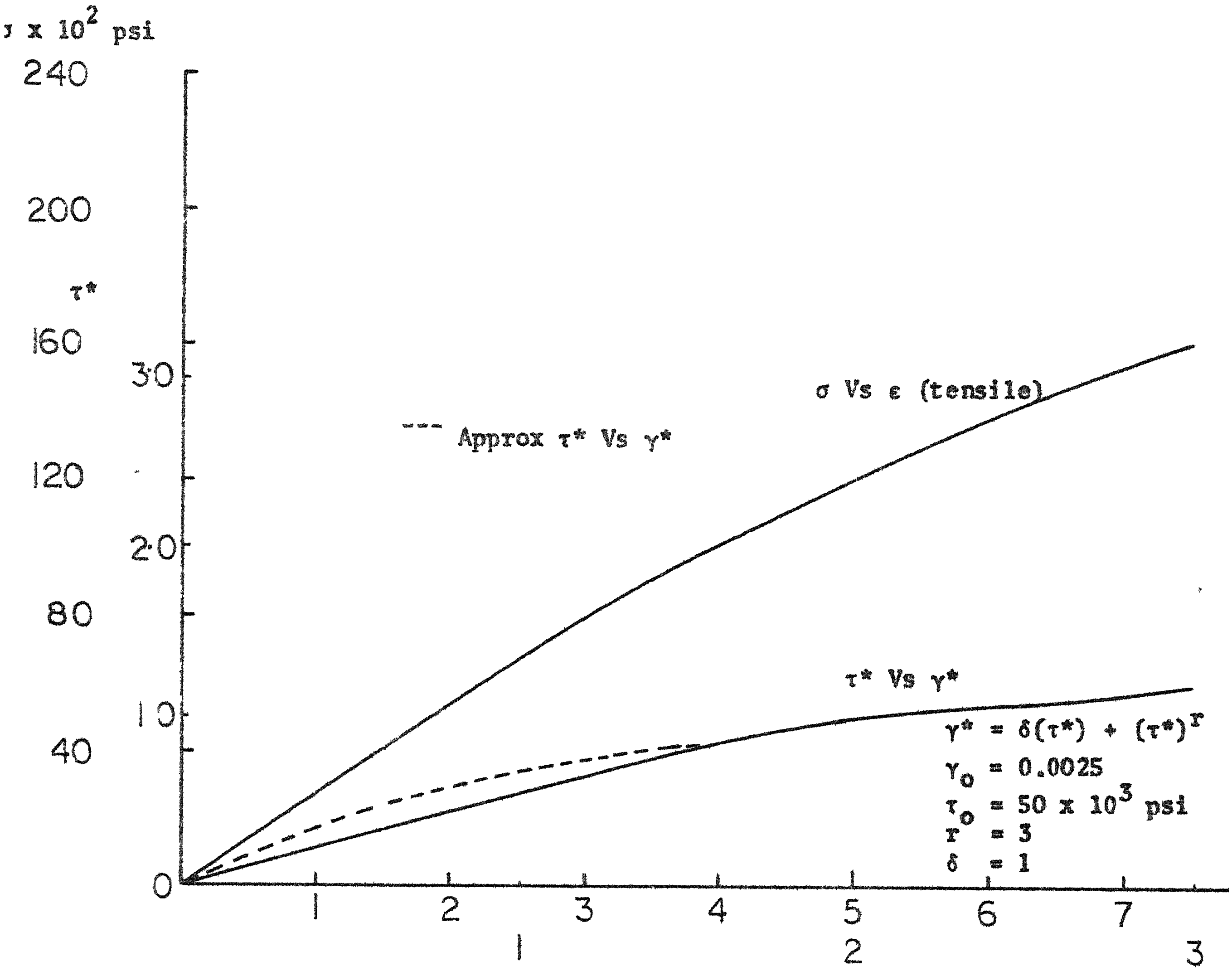
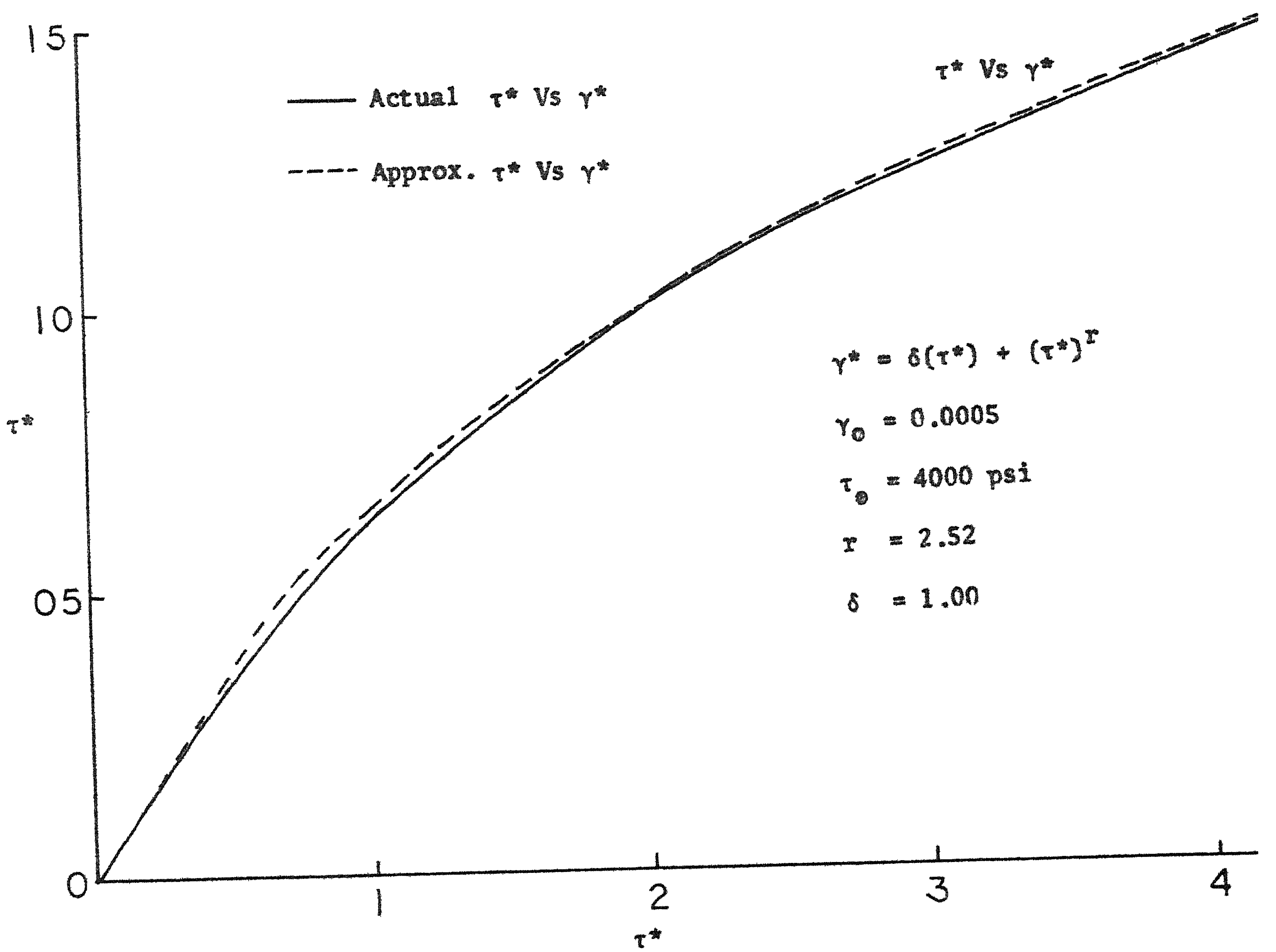

σ x 10² psi
240
200
τ*
160
3·0
120
2·0
80
1·0
40
0
σ Vs ε (tensile)
--- Approx τ* Vs γ*
τ* Vs γ*
γ* = δ(τ*) + (τ*)^r
γ₀ = 0.0025
τ₀ = 50 x 10³ psi
r = 3
δ = 1
1
2
3
4
5
6
7
1
2
3

Fig. 5



15
10
05
0
τ^*
1
2
3
4
τ^*
Actual τ^* Vs γ^*
Approx. τ^* Vs γ^*
τ^* Vs γ^*
$\gamma^* = \delta(\tau^*) + (\tau^*)^r$
$\gamma_o = 0.0005$
$\tau_o = 4000$ psi
$r = 2.52$
$\delta = 1.00$

Fig. 6

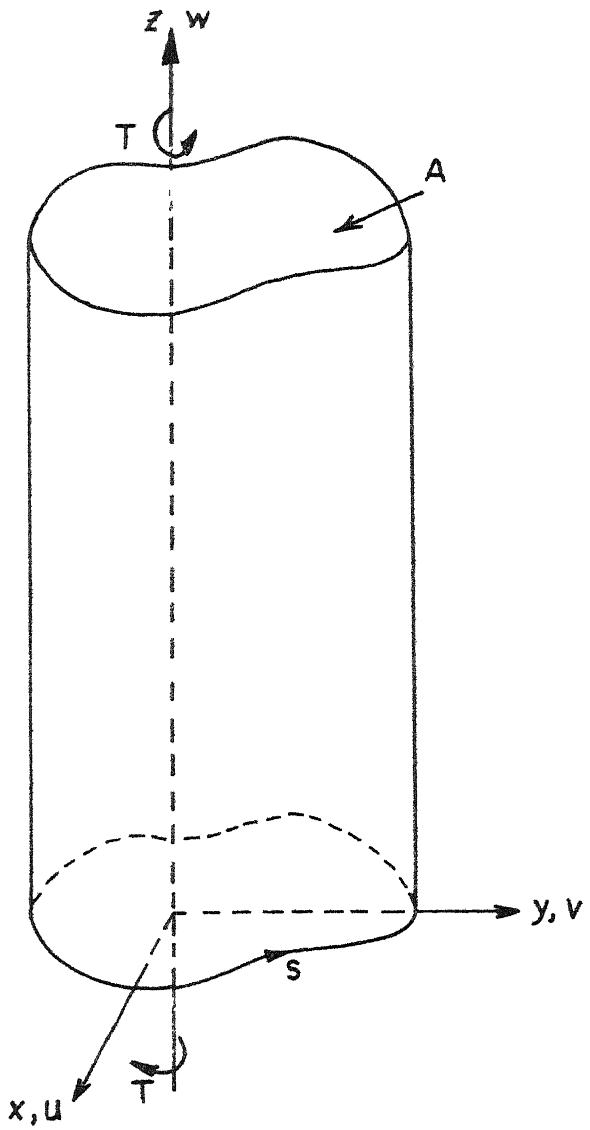


Figure. 7

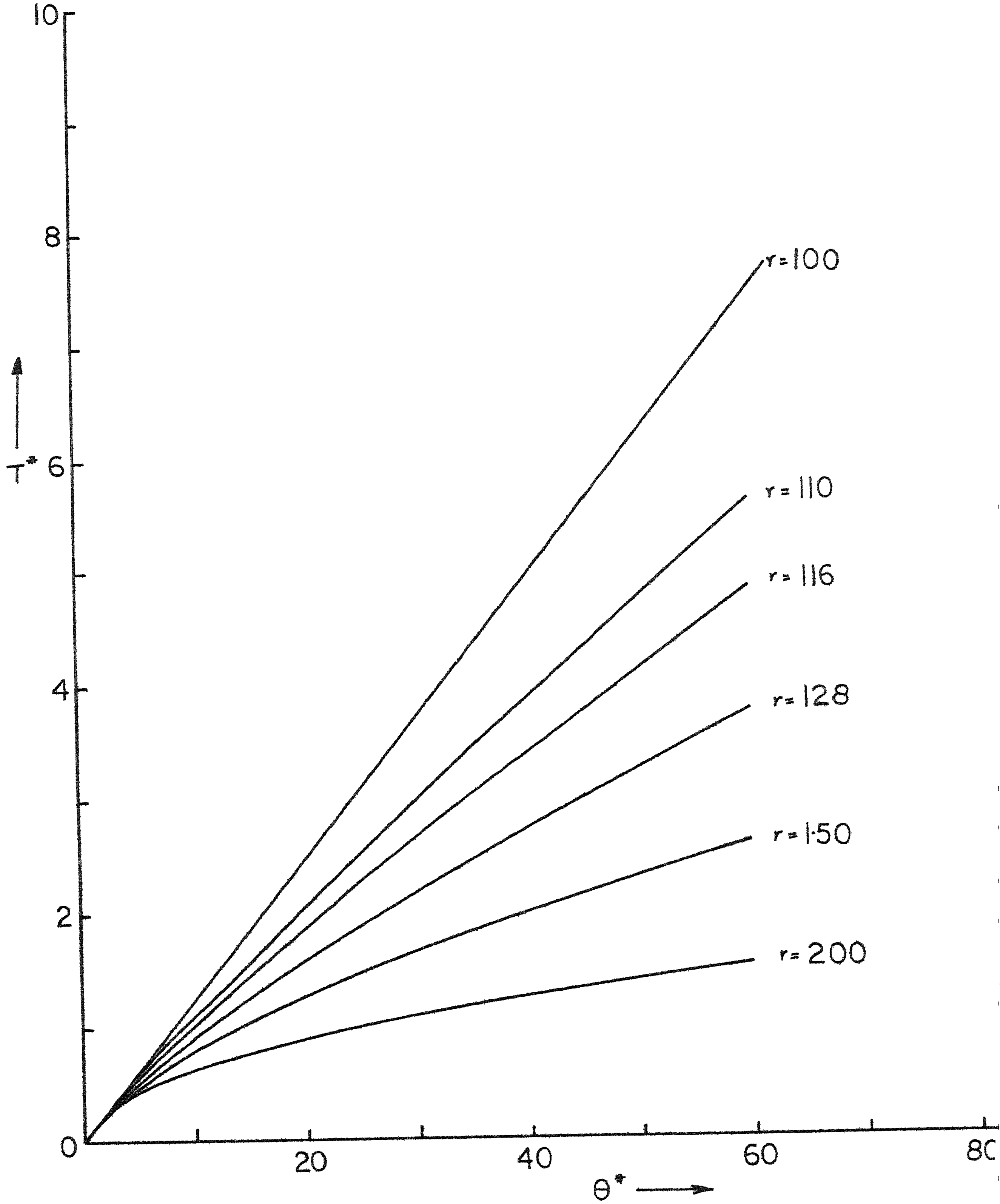


Fig. 8

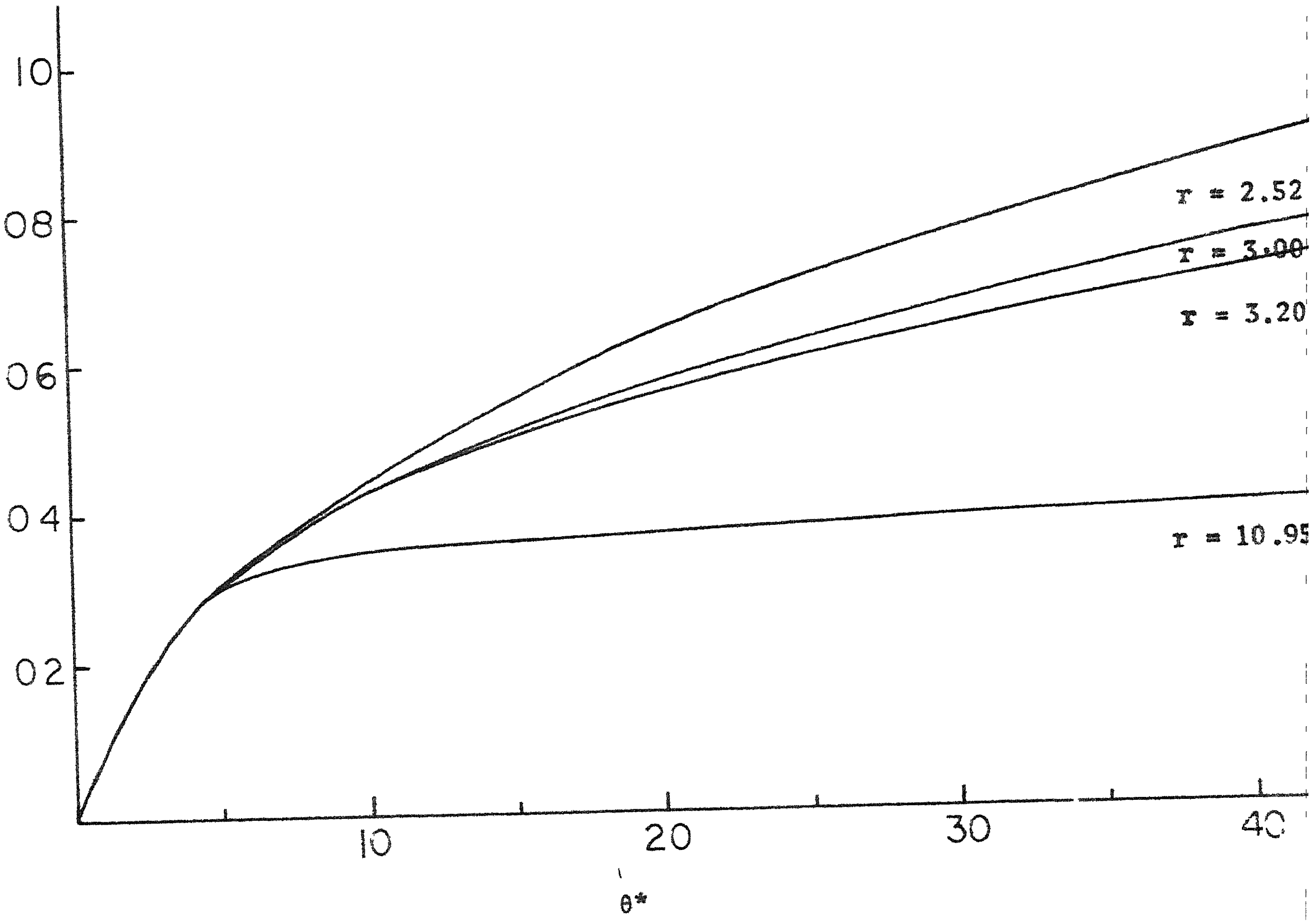


Fig. 9

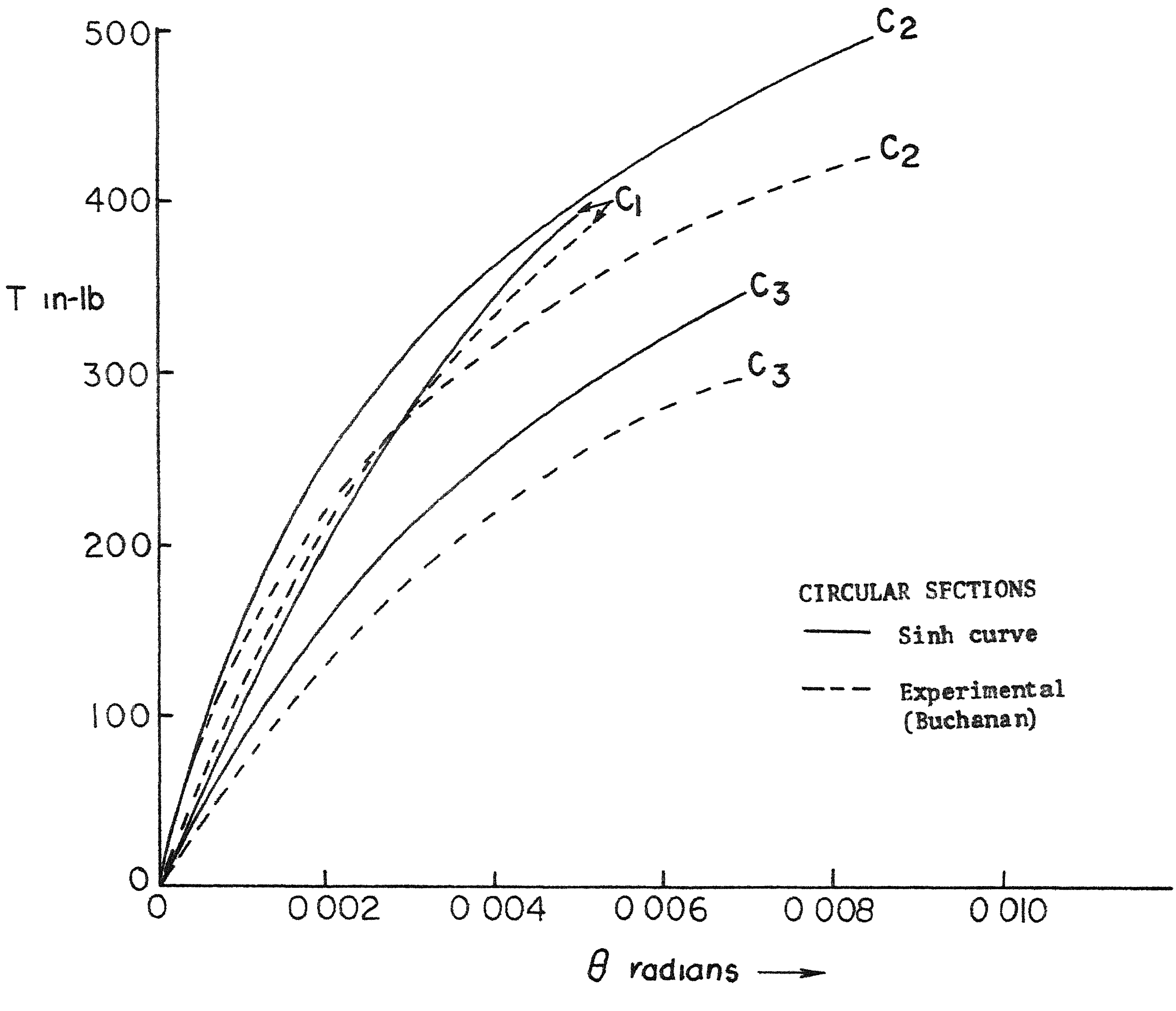

C2
C2
C1
C3
C3
500
400
300
200
100
0
T in-lb
0
0 002
0 004
0 006
0 008
0 010
θ radians
CIRCULAR SECTIONS
Sinh curve
Experimental
(Buchanan)

Figure. 10

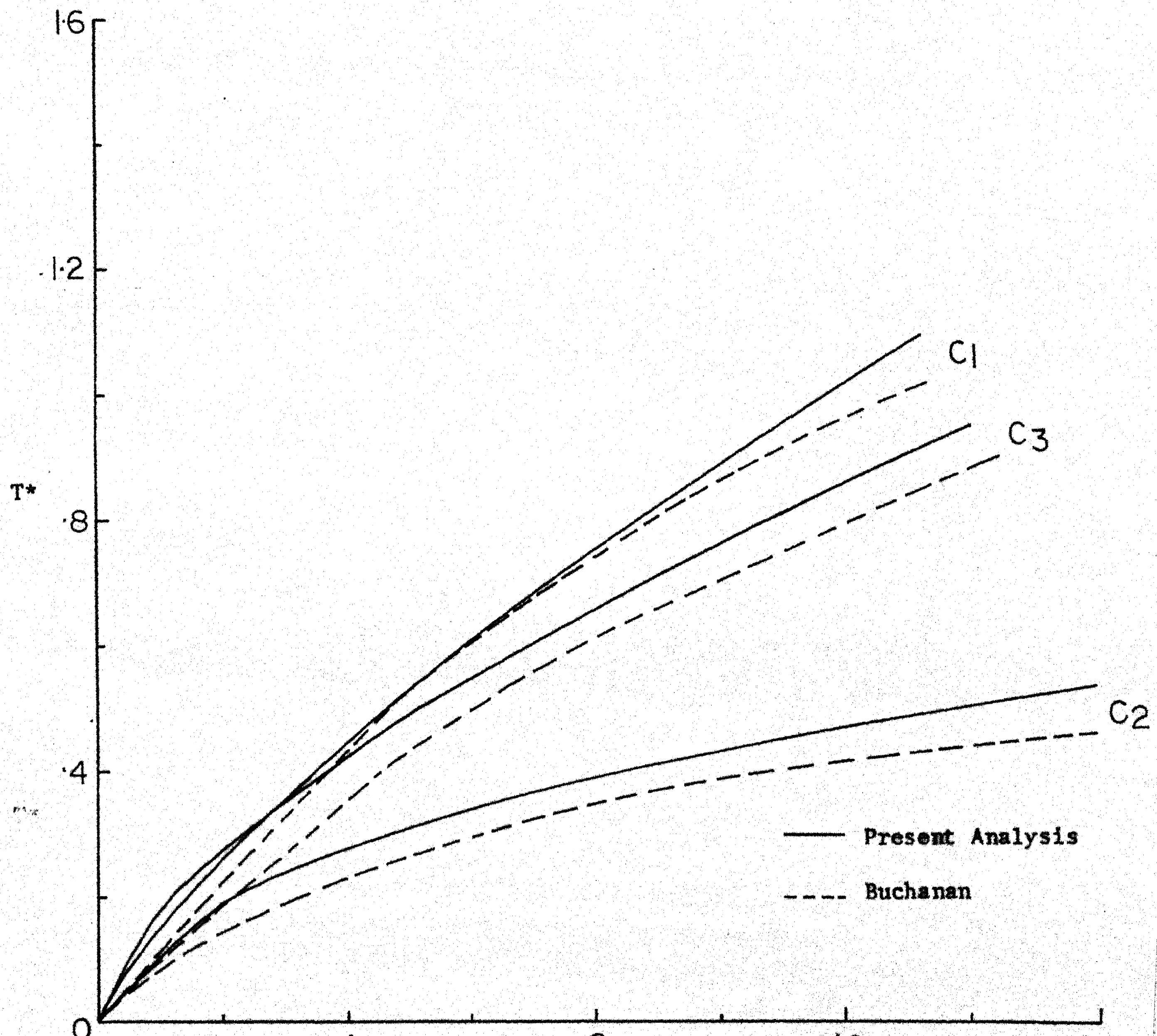


Fig. 11

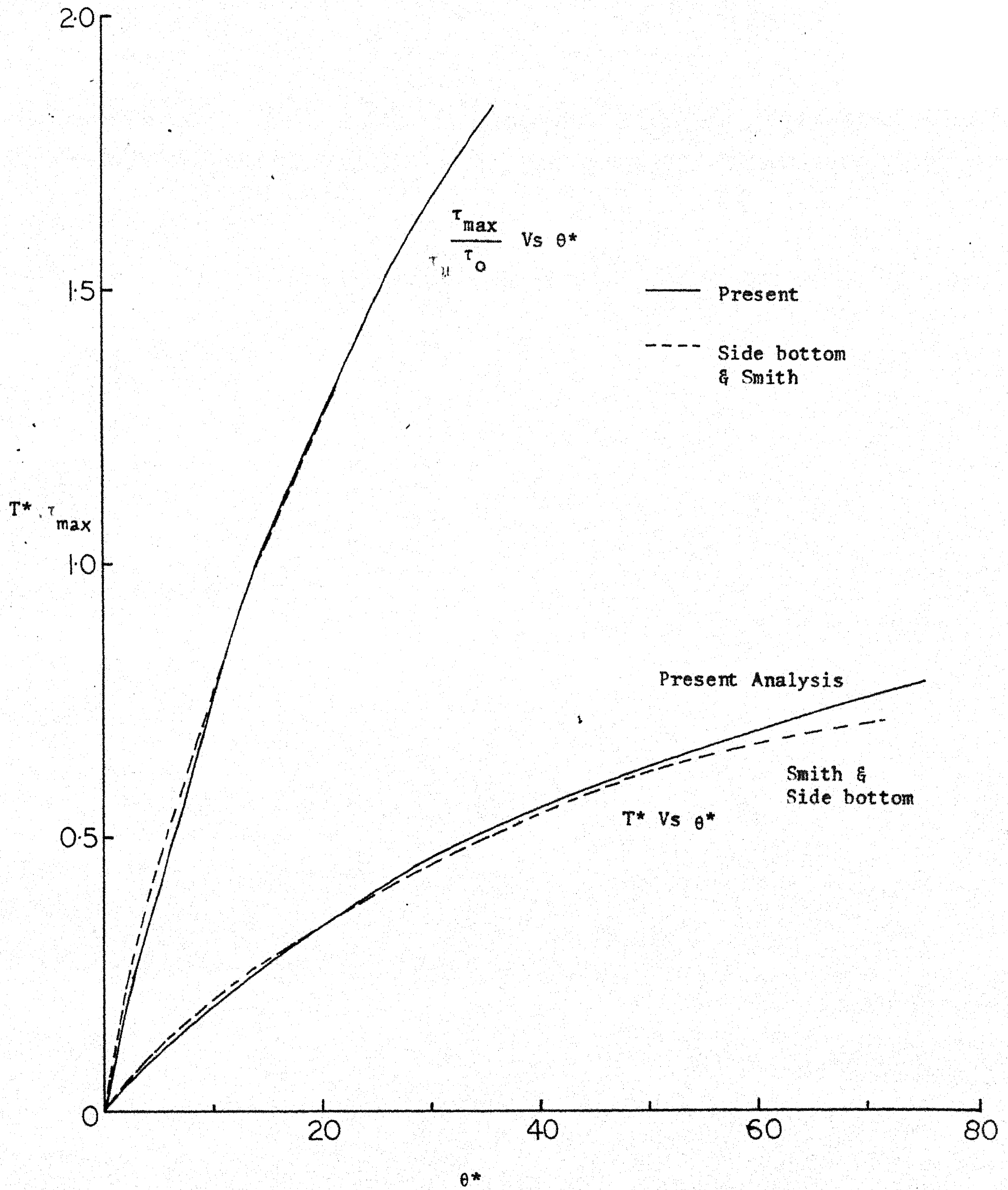

2·0
1·5
1·0
0·5
0
20
40
60
80
$\frac{\tau_{max}}{\tau_o}$ Vs θ*
Present
Side bottom
& Smith
T*, τ_{max}
Present Analysis
Smith &
Side bottom
T* Vs θ*
θ*

Fig. 12

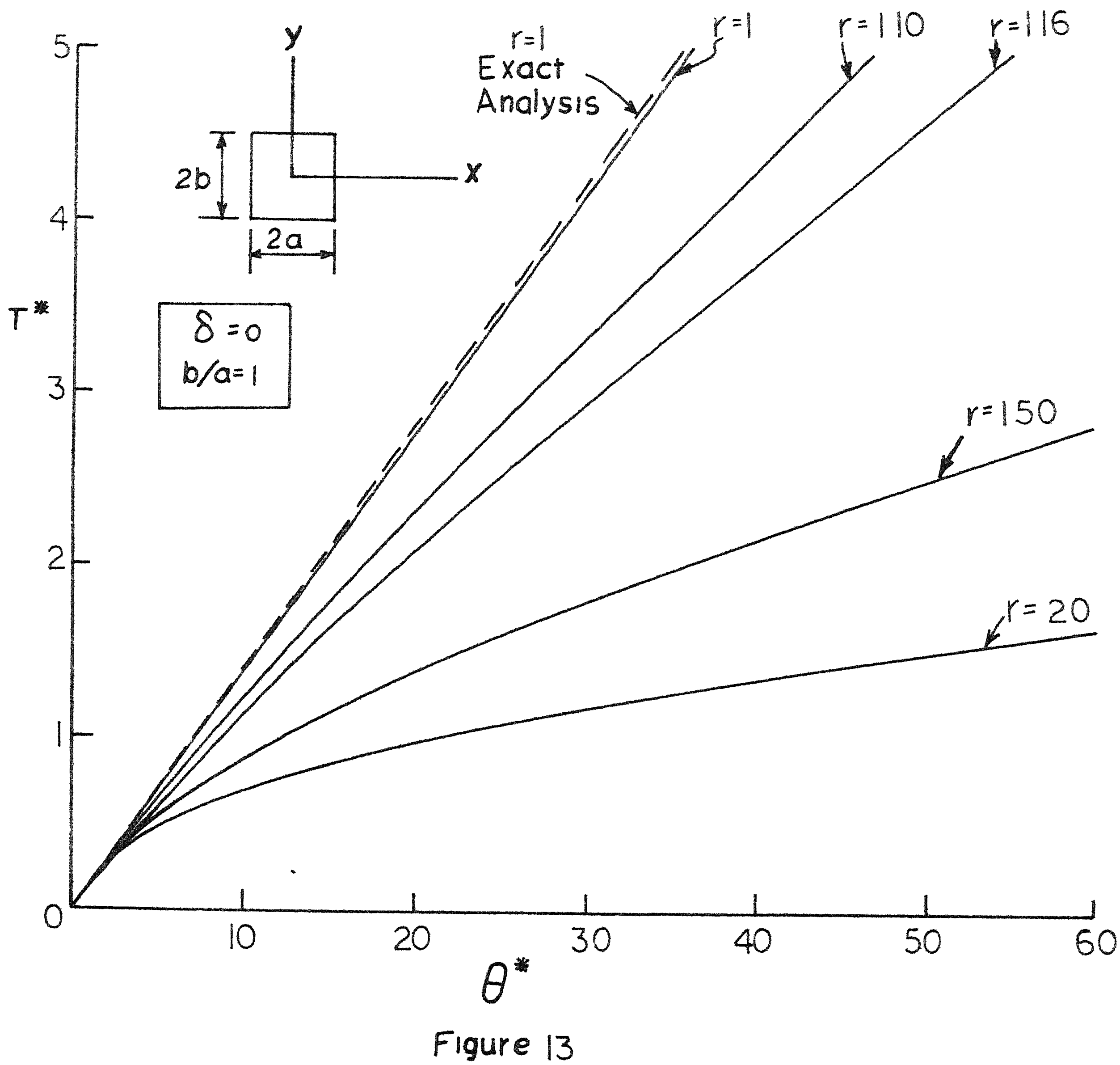
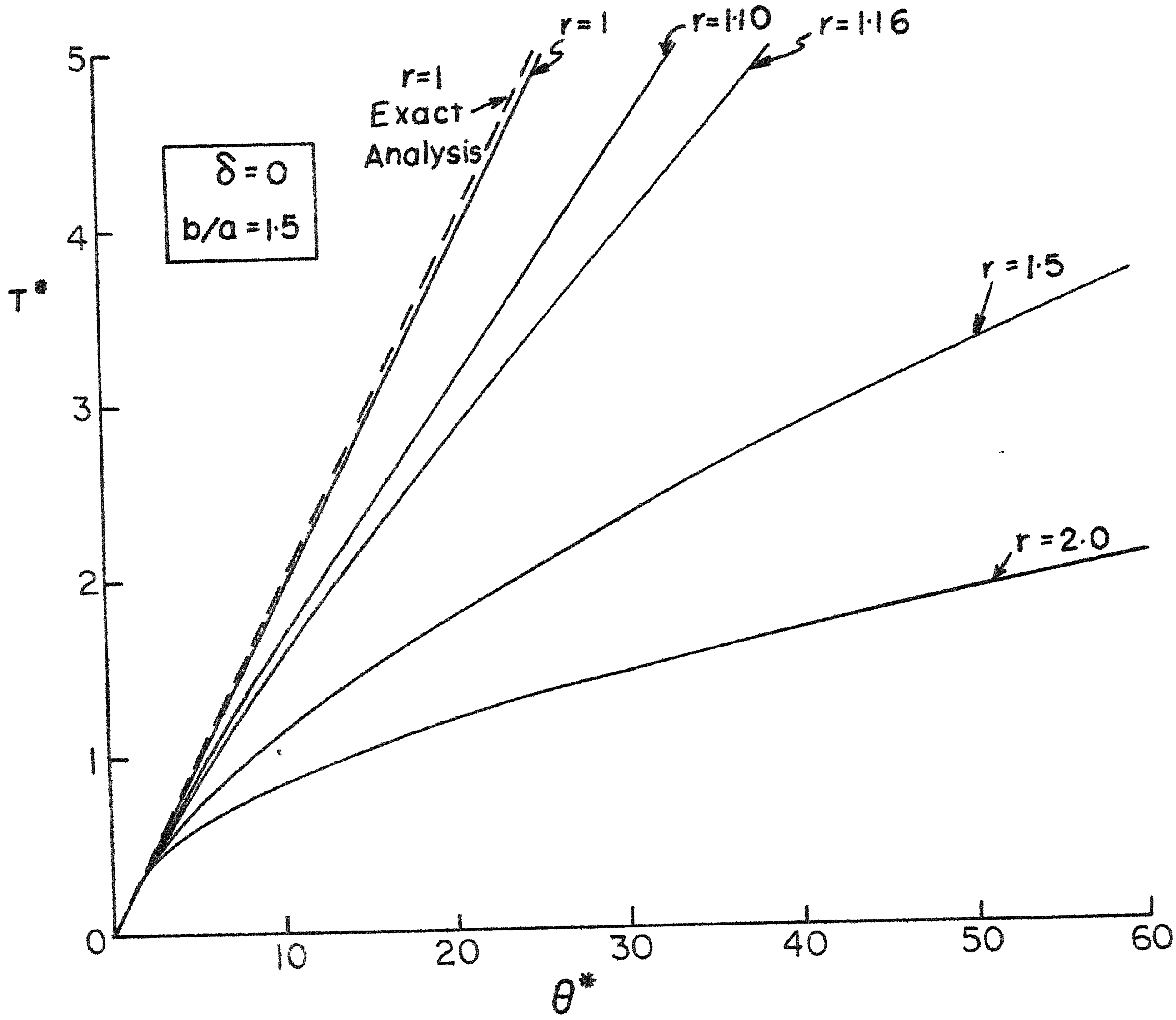
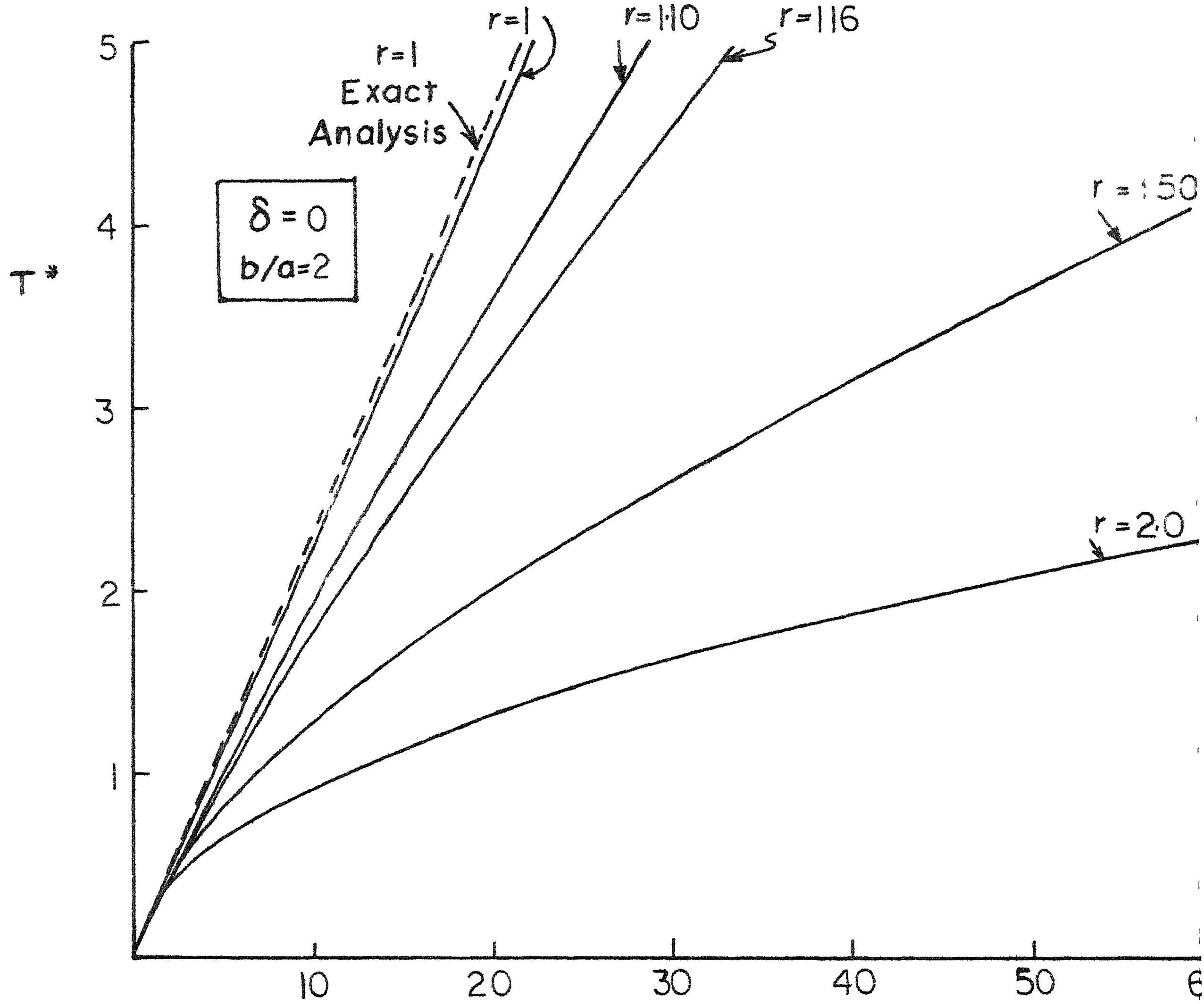

r=1
Exact
Analysis
r=1
r=110
r=116
r=150
r=20
y
x
2b
2a
δ=0
b/a=1
T*
θ*
0
1
2
3
4
5
10
20
30
40
50
60

Figure 13



r=1
r=1·10
r=1·16
r=1
Exact
Analysis
δ=0
b/a=1·5
r=1·5
r=2·0
T*
θ*
0
1
2
3
4
5
10
20
30
40
50
60

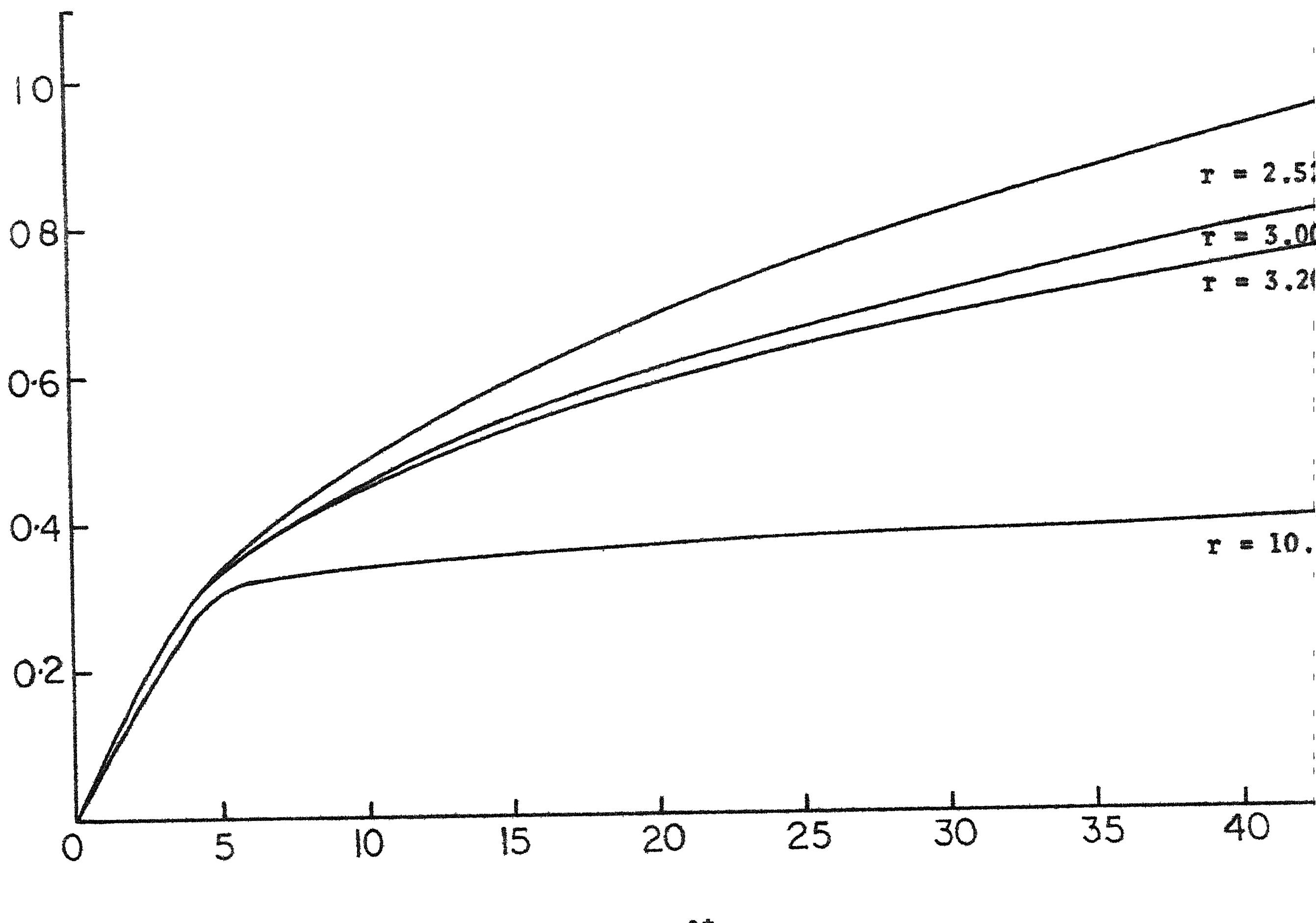
Figure. 14



r=1
r=1
Exact
Analysis
r=1.10
r=1.16
r=1.50
r=2.0
δ=0
b/a=2
T*
θ*
5
4
3
2
1
10
20
30
40
50

Figure 15

b/a = 1.00



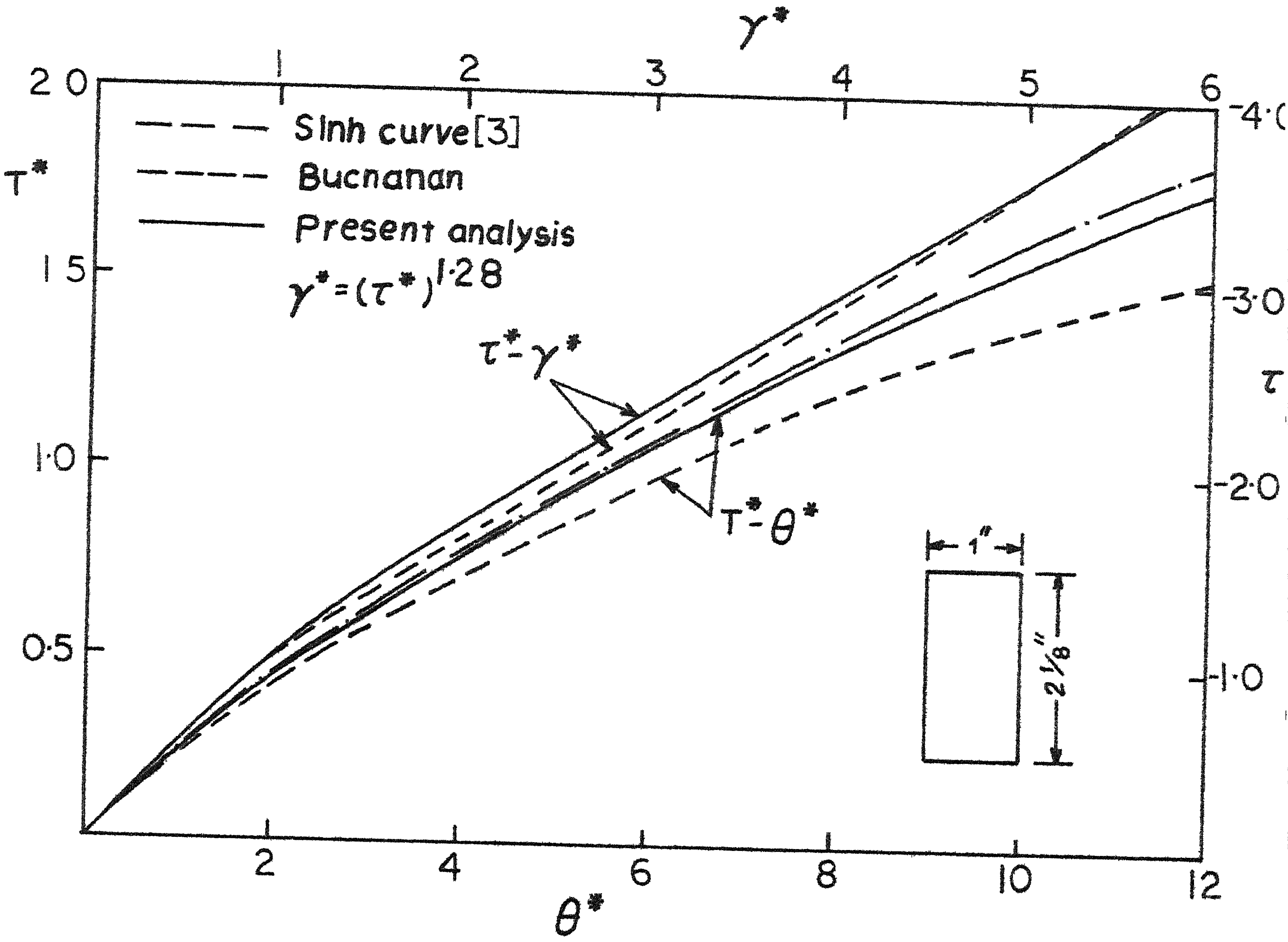

γ*
Sinh curve[3]
Bucnanan
Present analysis
γ*=(τ*)^1.28
τ*-γ*
T*-θ*
T*
θ*
1"
2 1/8"
2 0
1 5
1·0
0·5
2
4
6
8
10
12
1
3
5
-4·0
-3·0
-2·0
-1·0

Figure. 17

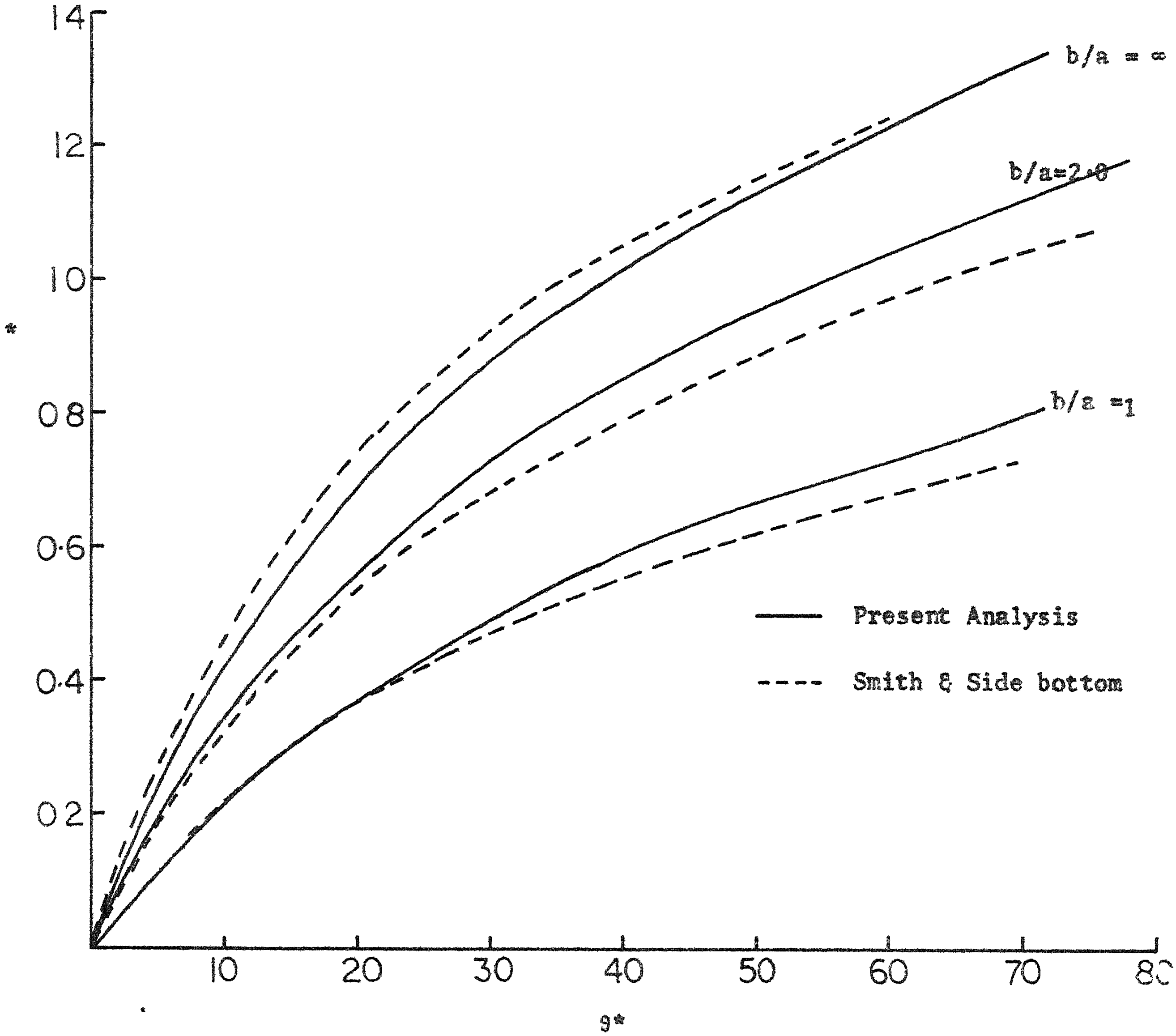
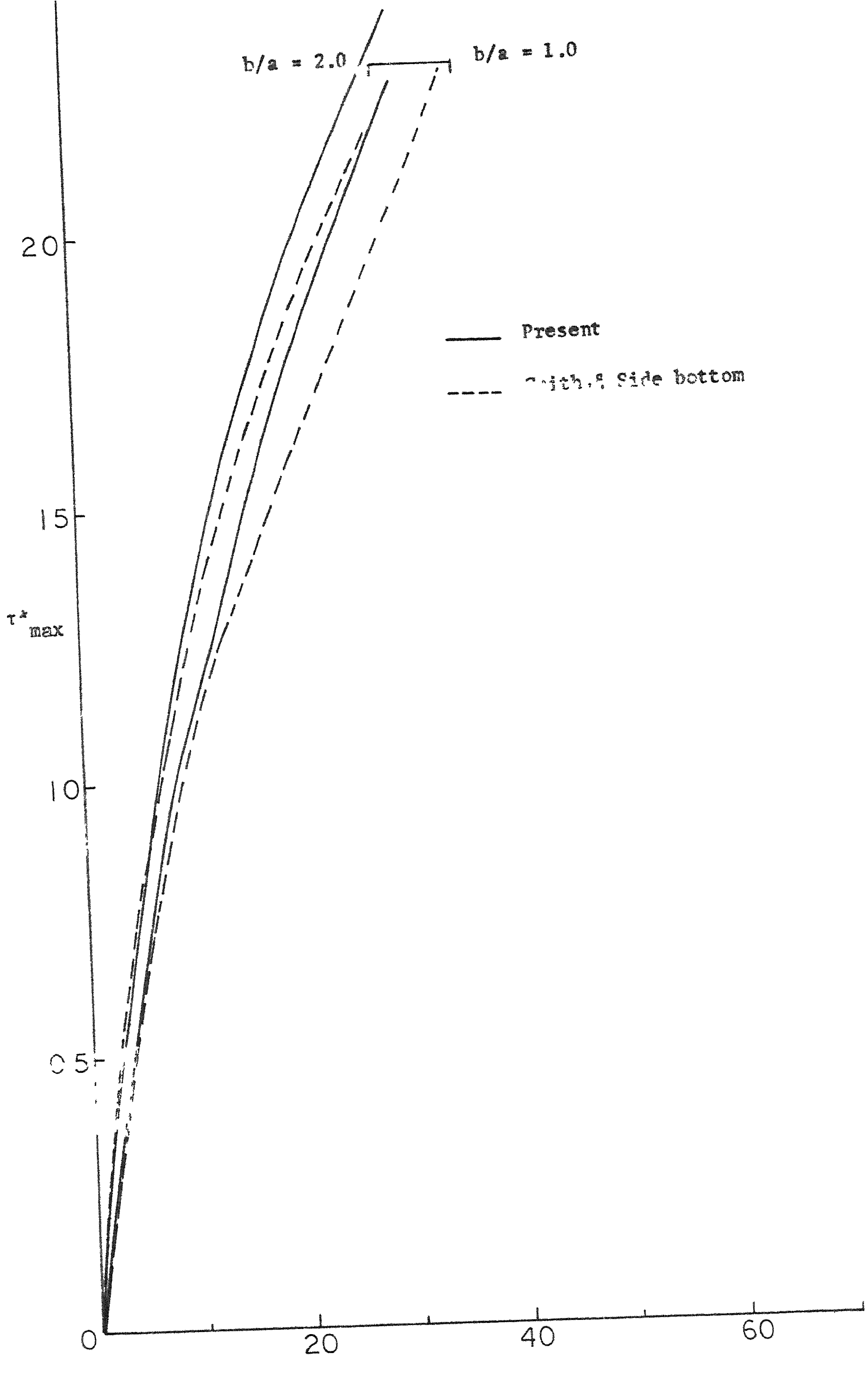


Fig. 18



b/a = 2.0
b/a = 1.0
Present
Smith's Side bottom
20
15
10
0.5
0
20
40
60
τ*max
θ*

Table 4. T* for Various Values of θ^* for Circular Cross-section

θ^*	$\delta = 0$						$\delta = 1$				Elastic	Plastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95		
0	0	0	0	0	0	0	0	0	0	0	0	
5	0.625	0.588	0.507	0.541	0.502	0.450	0.316	0.316	0.316	0.316	0.625	
10	1.250	1.105	1.037	0.929	0.796	0.636	0.460	0.438	0.431	0.351	1.250	
15	1.875	1.598	1.470	1.275	1.043	0.779	0.562	0.520	0.506	0.368	1.875	
20	2.500	2.075	1.884	1.597	1.264	0.900	0.646	0.585	0.565	0.380	2.500	
25	3.125	2.541	2.284	1.902	1.467	1.006	0.717	0.638	0.613	0.389	3.125	
30	3.750	3.000	2.672	2.192	1.657	1.102	0.779	0.684	0.654	0.396	3.750	0.333
35	4.375	3.451	3.053	2.473	1.836	1.190	0.836	0.725	0.691	0.402	4.375	
40	5.000	3.900	3.425	2.745	2.006	1.273	0.887	0.762	0.725	0.408	5.000	
45	5.625	4.337	3.791	3.010	2.171	1.350	0.935	0.796	0.755	0.412	5.625	
50	6.250	4.773	4.151	3.268	2.330	1.423	0.979	0.828	0.783	0.417	6.250	
55	6.875	5.205	4.507	3.520	2.481	1.493	1.021	0.857	0.809	0.420	6.875	
60	7.500	5.632	4.858	3.768	2.630	1.559	1.060	0.885	0.833	0.424	7.500	

Table 5 : τ^*_{max} for various Values of θ^* for Circular Cross-section

θ^*	$\delta = 0$						$\delta = 1$				Elastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95	
0	0	0	0	0	0	0	0	0	0	0	0
5	2.500	2.280	2.176	2.004	1.782	1.501	1.094	1.071	1.064	0.976	2.500
10	5.000	4.284	3.956	3.446	2.828	2.124	1.551	1.144	1.404	1.048	5.000
15	7.494	6.194	5.610	4.728	3.708	2.598	1.878	1.689	1.630	1.091	7.500
20	9.994	8.048	7.248	6.120	4.492	2.922	2.142	1.885	1.806	1.122	10.000
25	12.491	9.856	8.716	7.048	5.212	3.356	2.367	2.049	1.951	1.146	12.500
30	14.998	11.632	10.306	8.128	5.886	3.676	2.565	2.191	2.077	1.166	15.000
35	17.487	13.384	11.646	9.168	6.522	3.972	2.744	2.317	2.189	1.183	17.500
40	19.985	15.110	13.066	10.174	7.130	4.244	2.907	2.431	2.290	1.198	20.000
45	22.483	16.818	14.464	12.114	7.714	4.504	3.058	2.536	2.382	1.211	22.500
50	24.981	18.508	15.840	13.048	8.274	4.746	3.199	2.633	2.467	1.223	25.000
55	27.480	20.184	17.196	13.968	8.818	4.978	3.332	2.724	2.547	1.234	27.500
60	30.018	21.846	18.536	14.870	9.344	5.200	3.458	2.809	2.621	1.244	30.000

Table 6 : Γ* for Various Values of θ* for b/a = 1.0

θ*	δ = 0						δ = 1				Elastic	Plastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95		
0	0	0	0	0	0	0	0	0	0	0	0	
5	0.701	0.655	0.632	0.594	0.543	0.476	0.336	0.332	0.331	0.305	0.703	
10	1.403	1.229	1.148	1.020	0.863	0.674	0.485	0.457	0.447	0.336	1.406	
15	2.105	1.777	1.628	1.401	1.130	0.826	0.592	0.541	0.523	0.351	2.309	
20	2.807	2.309	2.087	1.753	1.369	0.953	0.679	0.606	0.583	0.363	2.812	
25	3.509	2.828	2.530	2.087	1.589	1.066	0.752	0.661	0.632	0.371	3.155	
30	4.211	3.338	2.960	2.407	1.794	1.167	0.817	0.708	0.674	0.378	4.218	0.333
35	4.913	3.840	3.381	2.715	1.988	1.261	0.875	0.750	0.712	0.384	4.921	
40	5.615	4.336	3.793	3.014	2.174	1.348	0.929	0.788	0.745	0.390	5.624	
45	6.317	4.826	4.199	3.304	2.351	1.430	0.979	0.823	0.776	0.394	6.327	
50	7.019	5.312	4.598	3.588	2.522	1.507	1.025	0.855	0.805	0.398	7.030	
55	7.721	5.792	4.992	3.865	2.688	1.580	1.068	0.885	0.831	0.401	7.733	
60	8.423	6.269	5.380	4.137	2.849	1.651	1.109	0.914	0.856	0.405	8.436	

Table 7 : T* for Various Values of θ* for b/a = 2.0

θ*	δ = 0						δ = 1				Elastic	Plastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95		
0	0	0	0	0	0	0	0	0	0	0	0	
5	1.141	1.038	0.989	0.909	0.806	0.676	0.483	0.468	0.464	0.389	1.145	
10	2.283	1.951	1.798	1.563	1.280	0.956	0.687	0.633	0.616	0.423	2.290	
15	3.426	2.820	2.550	2.145	1.677	1.171	0.834	0.745	0.717	0.442	3.440	
20	4.568	3.663	3.268	2.686	2.032	1.352	0.952	0.832	0.795	0.454	4.580	
25	5.709	4.486	3.961	3.197	2.358	1.511	1.052	0.906	0.861	0.465	5.722	
30	6.851	5.295	4.635	3.686	2.663	1.656	1.142	0.969	0.917	0.474	6.870	0.412
35	7.993	6.092	5.294	4.158	2.951	1.788	1.222	1.026	0.967	0.480	8.015	
40	9.135	6.879	5.940	4.615	3.226	1.912	1.295	1.077	1.012	0.487	9.160	
45	10.277	7.656	6.575	5.060	3.489	2.028	1.363	1.124	1.053	0.493	10.305	
50	11.419	8.426	7.200	5.494	3.743	2.138	1.426	1.167	1.094	0.497	11.450	
55	12.561	9.188	7.817	5.919	3.989	2.242	1.486	1.208	1.127	0.502	12.595	
60	13.703	9.945	8.425	6.336	4.227	2.342	1.542	1.246	1.160	0.506	13.740	

Table 8: τ^*_{max} for Various Values of θ^* for $b/a = 1.0$

θ^*	$\delta = 0$						$\delta=1$				Elastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95	
0	0	0	0	0	0	0	0	0	0	0	0
5	3.514	3.212	3.066	2.825	2.510	2.098	1.510	1.464	1.447	1.172	3.375
10	7.028	6.032	5.570	4.856	3.986	2.967	2.142	1.966	1.908	1.254	6.750
15	10.542	8.722	7.902	6.665	5.222	3.633	2.594	2.306	2.215	1.303	9.825
20	14.056	11.328	10.128	8.345	6.326	4.194	2.958	2.573	2.453	1.339	13.500
25	17.570	13.874	12.276	9.934	7.340	4.691	3.268	2.796	2.650	1.363	16.870
30	21.085	16.376	14.364	11.455	8.290	5.136	3.542	2.990	2.822	1.391	20.250
35	24.599	18.842	16.406	12.921	9.186	5.138	3.789	3.162	2.973	1.411	23.625
40	28.113	21.272	18.406	14.342	10.042	5.933	4.015	3.318	3.110	1.428	27.000
45	31.627	23.678	20.374	15.725	10.863	6.292	4.223	3.460	3.235	1.444	30.375
50	35.141	26.058	22.312	17.074	11.653	6.633	4.418	3.593	3.350	1.458	33.750
55	38.654	28.414	24.222	18.394	12.417	6.957	4.602	3.717	3.458	1.471	37.125
60	42.169	30.754	26.103	19.687	13.159	7.268	4.775	3.833	3.560	1.483	40.500

Table 9 : τ^*_{max} for Various Values of θ^* for b/a = 2.0

θ^*	$\delta = 0$						$\delta = 1$				Elastic
	r=1.00	1.10	1.16	1.28	1.50	2.00	r=2.52	3.00	3.20	10.95	
0	0	0	0	0	0	0	0	0	0	0	0
5	4.754	4.317	3.846	3.384	2.809	2.120	1.556	1.446	1.409	1.051	4.650
10	9.508	7.769	6.990	5.816	4.608	2.992	2.146	1.887	1.806	1.118	9.300
15	14.261	11.232	9.912	7.984	5.844	3.677	2.568	2.189	2.076	1.160	13.950
20	19.016	14.592	12.712	9.996	7.088	4.241	2.909	2.428	2.285	1.190	18.600
25	23.769	17.872	15.400	11.900	8.208	4.741	3.201	2.628	2.461	1.216	23.250
30	28.523	21.088	18.024	13.721	9.280	5.194	3.458	2.802	2.576	1.236	27.900
35	33.277	24.264	20.584	15.478	10.288	5.487	3.690	2.958	2.747	1.253	32.550
40	38.031	27.400	23.032	17.180	11.240	5.997	3.902	3.098	2.869	1.268	37.200
45	42.785	30.488	25.568	18.836	12.160	6.360	4.099	3.228	2.980	1.282	41.850
50	47.539	33.560	27.992	20.452	13.040	6.704	4.283	3.347	3.084	1.294	46.500
55	52.292	36.592	30.390	22.033	13.896	7.032	4.456	3.459	3.180	1.306	51.15
60	57.047	39.608	32 757	23.583	14.728	7.344	4.620	3.564	3.270	1.316	55.800

b/a	1	1.5	2	3	4	∞
Present analysis	.254	.303	.317	.344	.356	.395
Plastic	.333	.393	.412	.445	.46	.5
Elastic	.208	.231	.246	.263	.281	.333

Table 10 : Values of K_1

Table 11 : Computed Ultimate Torques $\bar{T}$ vs. Experimental Values

Beam	Cross section in x in 2a x 2b	Experimental f'_t psi	Computed f'_t psi	$\bar{T}$ in kips (Experimental)	$\bar{T}$(in kips) (Computed)	$\frac{\bar{T}_{exp.}}{\bar{T}_{comp.}}$
Hsu [13]						
A1	10 x 15	354*		162	161	1.085
A2	10 x 15	354*		169	161	1.05
A3	10 x 10	367		102	93.5	1.09
A4	10 x 10	367		100	93.5	1.045
A5	10 x 20	343		216	216	1.00
A6	10 x 20	346		216	215	1.005
A7	6 x 11	354*		54	43.8	1.23
A8	6 x 11	354*		56.5	43.8	1.29
A9	6 x 19.5	361		101	88.2	1.145
A10	6 x 19.5	338		85	83.0	1.025
Navaratnarajah [35]						
4RW-4	4 x 4		485	9.05	7.9	1.19
4RT-4	4 x 4		544	10.26	8.85	1.16
6RT-4	4 x 6		485	16.601	14.10	1.18
6RW-4	4 x 6.125		485	17.194	14.90	1.15
Collins etal [38]						
W5	5 x 10.2		373	31.9	30.4	1.05
RP2	5.2 x 10.3		392	35.6	34.8	1.02

* Average value measured from tests [13].

8. SUMMARY AND CONCLUSIONS

In this dissertation, a method based on complementary energy have been presented for obtaining the torque Vs rotation and shear stress Vs rotation curves for prismatic members. The method can be applied to prismatic members of any arbitrary cross-section with isotropic, homogeneous material properties, provided shear stress strain curve of the material is known.

Shear stress strain curves for structural materials at present are obtained either from experimental torque Vs rotation curves for circular beams or can be obtained from tesnile stress strain curves of the material. But none of these methods give curves which represent actual shear stress strain curve in pure uniform shear. More research need to be undertaken for obtaining these curves in pure shear directly to replace the present methods. For this, modifications in the recentlyy developed technique of obtaining shearing strength of concrete in pure shear can also be considered.

Idealisation of shear stress strain curves is done by generalised Ramberg Osgood, hyperbolic Sinh and polynomial functions. Ramberg Osgood function is found to cover the widest range of material behaviour and hence used exclusively. However, there is a limit beyond which curve fitting can not be improved. Further attempts lead to spoilling of previous fitting in either the initial

stages or in the final stages of the curve. The possibility of using two curves can be thought of as the possible answer. The other alternative is to findout some other function more suited to any particular material and then extend the present technique for that function.

Results for prismatic members of circular and rectangular cross-sections have been presented for materials like concrete, aluminium, steel magnisium and reinforced plastics. The number of terms in the stress functions have been kept to two or three only. More no. of terms complicates the analysis and are not practicable. Membrane and sand tieap analogies suggest the possibility of making stress function to depend upon material properties as well besides satisfying the boundary conditions.

Comparison of results with the limited experiemental and analytical results show good agreement. It is observed that the Sinh curve for concrete gives value of ultimate torque which are in agreement with experimental results for rectangular sections. More knowledge on the stress-strain curves in uniaxial tension for different concrete is needed for getting the value of twist at at ultimate.

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APPENDIX

DIGITAL COMPUTER PROGRAM

A.1 Introduction

A digital computer program is written to handle numerical procedure outlined in section 5. The program consists of main program and one subroutine. It is quite general, and can be used to obtain dimensionless torque and dimensionless shear stress corresponding to any value of dimensionless rotation. The program covers almost all structural materials provided suitable values of 'r' and ' ' are given. For the understanding of overall logic and linkage of the program, the flow chart is given. Program for rectangular cross-section is given at the end. For circular cross-section, a similar program will work. The terms for input and output variables used in the program are given below. Other symbolic names used in the program are self explanatory.

A.2 Definition of Variables

(a) Input Variables

A,B,C = Values of the constants to start the iteration.

RAT(I),RATIO= b/a ratio for rectangular section

EMM(J),EM = Value of m ($= \frac{r+1}{2}$)

ND = Number of divisions for numerical integration

DELTA = Value of ' '

(b) Output Variables

Variables which are same as input variables are not defined here, the remaining variables are as follows

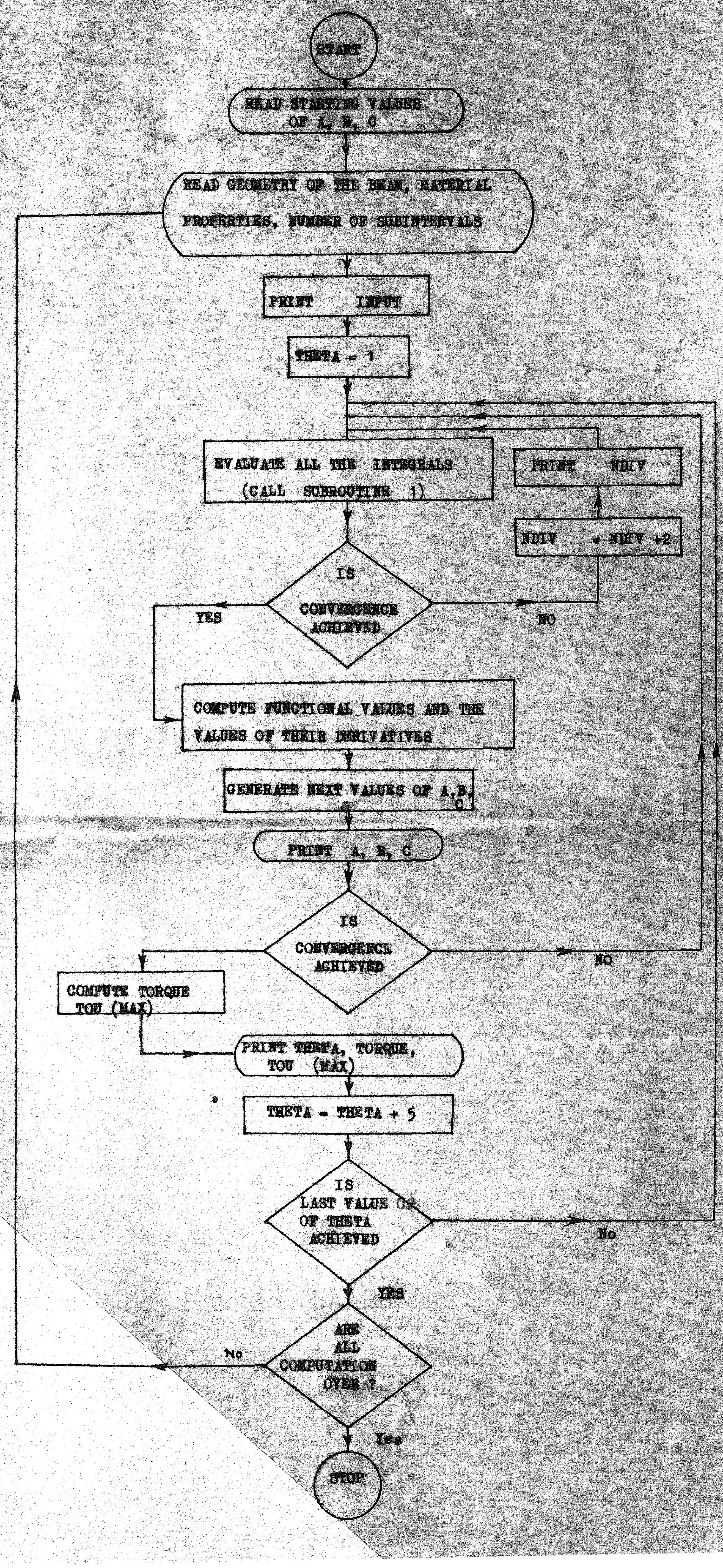
ITRN = Number of Itration

THETA =Dimensionless twist

Torque=Dimensionless Torque

TOU(Max)= Dimensionless Shrear Stress (maximum)

A.3 FLOW CHART FOR COMPUTER PROGRAM



```
C
C     TORSION OF RECTANGULAR CROSS-SECTION OF NON-LINEAR MATERIAL
C
      DIMENSION U(30),RAT(5),EMM(6)
      READ10,(RAT(I),I=1,5),(EMM(J),J=1,6),ND
   10 FORMAT(1X,5F6.3,2X,6F5.2,2X,I3)
      READ11,A,B,C,DELTA
   11 FORMAT(1X,F10.4)
      PRINT12
   12 FORMAT(///40X,50H TORSION OF RECTANGULAR BAR OF
     1NONLINEAR MATERIAL/)
      DO24I1=1,5
      DO24J1=1,6
      RATIO=RAT(I1)
      R1=RATIO*RATIO
      EM=EMM(J1)
      PRINT13,RATIO,EM,DELTA
   13 FORMAT(37X,7H B/A = ,F6.3,15X,4HEM= ,F5.2,15X,7HDELTA= ,I3)
      EM11=EM-1.
C
C     GENRATION OF COEFFICIENT FOR SIMPSONS INTEGRATION FORMULA
C
      N1=ND+1
      DO14IX=2,N1,2
      U(IX)=4.00
      IF(IX.EQ.ND) GO TO 14
      U(IX+1)=2.00
   14 CONTINUE
      U(1)=1.00
      U(N1)=1.00
      DIV=ND
      SEGMT=1.00/DIV
      PRINT15
   15 FORMAT(/30X,5HTHETA,13X,4HITRN,15X,1HA,19X,1HB,19X,1HC/)
      DO23L=2,1001,5
      N=L-1
      S=N
      DO19M=1,20
C
C     INTEGRAL  EVALUATION
C
      X=0.00
      V=0.0
      VV=0.0
      VVV=0.0
      V11=0.0
      V12=0.0
      V21=0.0
```

```
      V22=0.0
      V31=0.0
      V32=0.0
      VV11=0.0
      VV12=0.0
      VV21=0.0
      VV22=0.0
      VV31=0.0
      VV32=0.0
      VVV11=0.0
      VVV12=0.0
      VVV21=0.0
      VVV22=0.0
      VVV31=0.0
      VVV32=0.0
      DO17I=1,M1
      X2=X*X
      X3=X2-1.
      X4=X3*X3
      X5=2.*X2-1.
      P=0.0
      Q=0.0
      R=0.0
      P11=0.0
      P12=0.0
      P21=0.0
      P22=0.0
      P31=0.0
      P32=0.0
      Q11=0.0
      Q12=0.0
      Q21=0.0
      Q22=0.0
      Q31=0.0
      Q32=0.0
      R11=0.0
      R12=0.0
      R21=0.0
      R22=0.0
      R31=0.0
      R32=0.0
      Y=0.0
      DO16J=1,N1
      Y2=Y*Y
      Y3=Y2-1.
      Y4=Y3*Y3
      Y5=2.*Y2-1.
      CALLSUB1(X,Y,A,B,C,X2,X3,X4,X5,Y2,Y3,Y4,Y5,EM,R1,F7,F8,F9,F10,
     1 F11,F12,F13,F14,F15,F16,F17,F18,F19,F20,F21,F22,F23,F24,F25,F2
     2F27)
      P=P+F7*U(J)
      Q=Q+F8*U(J)
      R=R+F9*U(J)
      P11=P11+F10*U(J)
      P12=P12+F11*U(J)
```

```
      P21=P21+F12*U(J)
      P22=P22+F13*U(J)
      P31=P31+F14*U(J)
      P32=P32+F15*U(J)
      Q11=Q11+F16*U(J)
      Q12=Q12+F17*U(J)
      Q21=Q21+F18*U(J)
      Q22=Q22+F19*U(J)
      Q31=Q31+F20*U(J)
      Q32=Q32+F21*U(J)
      R11=R11+F22*U(J)
      R12=R12+F23*U(J)
      R21=R21+F24*U(J)
      R22=R22+F25*U(J)
      R31=R31+F26*U(J)
      R32=R32+F27*U(J)
      Y=Y+SEGMT
16    CONTINUE
      V=V+P*U(I)
      VV=VV+Q*U(I)
      VVV=VVV+R*U(I)
      V11=V11+P11*U(I)
      V12=V12+P12*U(I)
      V21=V21+P21*U(I)
      V22=V22+P22*U(I)
      V31=V31+P31*U(I)
      V32=V32+P32*U(I)
      VV11=VV11+Q11*U(I)
      VV12=VV12+Q12*U(I)
      VV21=VV21+Q21*U(I)
      VV22=VV22+Q22*U(I)
      VV31=VV31+Q31*U(I)
      VV32=VV32+Q32*U(I)
      VVV11=VVV11+R11*U(I)
      VVV12=VVV12+R12*U(I)
      VVV21=VVV21+R21*U(I)
      VVV22=VVV22+R22*U(I)
      VVV31=VVV31+R31*U(I)
      VVV32=VVV32+R32*U(I)
      X=X+SEGMT
17    CONTINUE
      DD=SEGMT*SEGMT/9.
      V=V*DD
      VV=VV*DD
      VVV=VVV*DD
      V11=V11*DD
      V12=V12*DD
      V21=V21*DD
      V22=V22*DD
      V31=V31*DD
      V32=V32*DD
      VV11=VV11*DD
      VV12=VV12*DD
      VV21=VV21*DD
      VV22=VV22*DD
```

```
      VV31=VV31*DD
      VV32=VV32*DD
      VVV11=VVV11*DD
      VVV12=VVV12*DD
      VVV21=VVV21*DD
      VVV22=VVV22*DD
      VVV31=VVV31*DD
      VVV32=VVV32*DD
C
C     SOLUTION OF NONLINEAR EQUATIONS BY NEWTONS METHOD
C
      TEM1=4.**EM11
      TEM2=4.**EM
      T1=EMM1/2.
      TR12=R1**(EM-1.)
C
C     EVALUATION OF FUNTIONAL VALUES AND THEIR PARTIAL DERIVATIVES
C
      D=DELTA
      AF=((5.68888*(R1+1.)*A+(1.13778*R1+0.812699)*B
     1+R1*(1.13778+0.812699*R1)-C)*D/32.)+TEM1*TR12*V-S/9.
      AG=(((1.787936/R1+0.270900)*B+(1.13778*R1+0.812699)*A
     1-R1*(0.162540/(R1+1.)+C))*D/32.)+TEM1*TR12*VV-S/45.
      AH=(((1.787936+0.270900*R1)-R1*C+(1.13778+0.812699*R1)*A
     1+0.162540*(R1+1.)*B)*(D/32.)+TEM1*TR12*VVV-S/45.
      AFA=(5.68888*(R1+1.)*D/32.)
     1+TEM2*T1*TR12*V11+TEM1*TR12*V12
      AFB=((1.13778*R1+0.812699)*D/32.)
     1+TEM2*T1*TR12*V21+TEM1*TR12*V22
      AFC=(R1*(1.13778*R1+0.812699*R1)*D/32.)
     1+TEM2*T1*TR12*V31+TEM1*R1**EM*V32
      AGA=((1.13778*R1+0.812699)*D/32.)
     1 +TEM2*T1*TR12*VV11+TEM1*TR12*VV12
      AGB=((1.787936*R1+.270900)*D/32.)
     1 +TEM2*T1*TR12*VV21+TEM1*TR12*VV22
      AGC=(R1*(0.162540/(R1+1.))*D/32.)
     1 +TEM2*T1*R1**EM*VV31+TEM1*R1**EM*VV32
      AHA=((1.13778+0.812699*R1)*D/32.)
     1 +TEM2*T1*TR12*VVV11+TEM1*TR12*VVV12
      AHB=(0.162540*(R1+1.)*D/32.)
     1 +TEM2*T1*TR12*VVV21+TEM1*TR12*VVV22
      AHC=(R1*(1.787936+0.270900*R1)*D/32.)
     1 +TEM2*T1*R1**EM*VVV31+TEM1*R1**EM*VVV32
C
C     EVALUATION OF VALUES OF A,B,C
C
      PP =-AF*(AGB*AHC-AGC*AHB)+AG*(AFB*AHC-AHB*AFC)
     1-AH*(AFB*AGC-AGB*AFC)
      QQ=AFA*(-AG*AHC+AH*AGC)-AGA*(-AF*AHC+AH*AFC)
     1+AHA*(-AF*AGC+AG*AFC)
      RR=AFA*(-AH*AGB+AG*AHB)-AGA*(-AH*AFB+AF*AHB)
     1+AHA*(-AG*AFB+AF*ACB)
      AJ=AFA*(AGB*AHC-AHB*AGC)+AGA*(AHB*AFC-AFB*AHC)
     1+AHA*(AFB*AGC-AGB*AFC)
```

```
      PP=PP/AJ
      QQ=QQ/AJ
      RR=RR/AJ
      A=A+PP
      B=B+QQ
      C=C+RR
      PRINT18,N,M,A,B,C
   18 FORMAT(21X,I4,11X,I3,10X,E15.8,10X,E15.8,10X,E15.8)
      IF(ABS(PP).LT.1.E-6.AND.ABS(QQ).LT.1.E-6.AND.ABS(RR).LT.1.E-6)
     1GOTO20
   19 CONTINUE
   20 PRINT21
   21 FORMAT(/35X,5HTHETA,19X,6HTORQUE,23X,8HTOU(MAX)/)
      DT=4.*R1*(5.*A+3+R1*C)/45.
      DTOU=2.*R1*(A+B)
      PRINT22,N,DT,DTOU
   22 FORMAT(35X,I10,15X,F10.4,15X,F10.4)
   23 CONTINUE
   24 CONTINUE
      STOP
      END
C
C     SUBROUTINE COMPUTES VALUE OF INTEGRAND AT GRID PONTS
C
      SUBROUTINE SUB1(X,Y,A,B,C,X2,X3,X4,X5,Y2,Y3,Y4,Y5,EM,R1,F7,F8,
     1F9,F10,F11,F12,F13,F14,F15,F16,F17,F18,F19,F20,F21,F22,F23,F24,
     2F25,F26,F27)
      F1=R1*X2*Y4*(A+B*X5+R1*C*Y2)**2+Y2*X4*(A+B*X2+R1*C*Y5)**2
      IF(F1)1,10,3
    1 PRINT2,F1
    2 FORMAT(1X,E10.4)
      STOP
    3 IF(ABS(X-1.).LT.1.E-06.AND.ABS(Y-1.).LT.1.E-06)GOTO10
      IF(EM-1.)4,5,4
    4 F2=F1**(EM-1.)
      GOTO6
    5 F2=1.
    6 IF(EM-2.)7,8,7
    7 F3=F1**(EM-2.)
      GOTO9
    8 F3=1.
    9 F4=R1*X2*Y4*(A+B*X5+R1*C*Y2)
      F5=Y2*X4*(A+B*X2+R1*C*Y5)
      F6=F4+F5
      F7=F2*F6
      F8=F2*(F4*X5+F5*X2)
      F9=F2*(F4*Y2+F5*Y5)
      F10=F3*F6**2
      F11=F2*(R1*X2*Y4+Y2*X4)
      F12=F3*F6*(F4*X5+F5*X2)
      F13=F2*(R1*X2*Y4*X5+Y2*X4*X2)
      F14=F3*F6*(F4*Y2+F5*Y5)
      F15=F2*(R1*X2*Y4*Y2+Y2*X4*Y5)
      F16=F3*F6*(F4*X5+F5*X2)
```

```
      F17=F2*(R1*X2*Y4*X5+Y2*X4*X2)
      F18=F3*(F4*X5+F5*Y2)**2
      F19=F2*(R1*X2*Y4*X5*X5+Y2*X4*Y2*X2)
      F20=F3*(F4*Y2+F5*X5)*(F4*X5+F5*Y2)
      F21=F2*(R1*X2*Y4*Y5*Y2+Y2*Y4*Y5*Y2)
      F22=F3*(F4*Y2+F5*Y5)*F6
      F23=F2*(R1*X2*Y4*Y2+Y2*X4*Y5)
      F24=F3*(F4*Y5+F5*X2)*(F4*Y2+F5*Y5)
      F25=F2*(R1*X2*Y4*Y2*X5+Y2*X4*Y5*X2)
      F26=F3*(F4*Y2+F5*Y5)**2
      F27=F2*(R1*X2*Y4*Y2*Y2+Y2*X4*Y5*Y5)
      RETURN
  10  F7=0.0
      F8=0.0
      F9=0.0
      F10=0.0
      F11=0.0
      F12=0.0
      F13=0.0
      F14=0.0
      F15=0.0
      F16=0.0
      F17=0.0
      F18=0.0
      F19=0.0
      F20=0.0
      F21=0.0
      F22=0.0
      F23=0.0
      F24=0.0
      F25=0.0
      F26=0.0
      F27=0.0
      RETURN
      END
$ENTRY
```